

A new trace bilinear form on cyclic \mathbb{F}_q -linear \mathbb{F}_{q^t} -codes

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Abstract Let \mathbb{F}_q be a finite field of cardinality q , where q is a power of a prime number p , $t \geq 2$ an even number satisfying $t \not\equiv 1 \pmod{p}$ and \mathbb{F}_{q^t} an extension field of \mathbb{F}_q with degree t . First, a new trace bilinear form on $\mathbb{F}_{q^t}^n$ which is called Δ -bilinear form is given, where n is a positive integer coprime to q . Then according to this new trace bilinear form, bases and enumeration of cyclic Δ -self-orthogonal and cyclic Δ -self-dual \mathbb{F}_q -linear \mathbb{F}_{q^t} -codes are investigated when $t = 2$. Furthermore, some good \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes are obtained.

Keywords Δ -bilinear form · \mathbb{F}_q -linear \mathbb{F}_{q^t} -codes · cyclic Δ -self-orthogonal \mathbb{F}_q -linear \mathbb{F}_{q^t} -codes · cyclic Δ -self-dual \mathbb{F}_q -linear \mathbb{F}_{q^t} -codes

1 Introduction

Additive codes over \mathbb{F}_4 were first introduced in 1998 in [2] connecting these codes to binary quantum codes. A year later, \mathbb{F}_p -linear codes over \mathbb{F}_{p^2} , where p is prime, were connected in [12] to nonbinary quantum codes. Additive codes were also generalized and studied in many papers, for example [1, 3, 7].

Let \mathbb{F}_q be a finite field of cardinality q , where q is a power of a prime number p , and \mathbb{F}_{q^t} be an extension field of \mathbb{F}_q with degree t . An additive code over \mathbb{F}_4 is simply an \mathbb{F}_2 -linear subspace of \mathbb{F}_4^n . A natural generalization of this is the following. An \mathbb{F}_q -linear \mathbb{F}_{q^t} -code \mathcal{C} of length n is an \mathbb{F}_q -linear subspace of $\mathbb{F}_{q^t}^n$. \mathcal{C}

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is said to be cyclic if $(c_{n-1}, c_0, c_1, \dots, c_{n-2}) \in \mathcal{C}$ for all $(c_0, c_1, \dots, c_{n-1}) \in \mathcal{C}$. In particular, \mathcal{C} is closed under componentwise addition and multiplication with elements from \mathbb{F}_q [6, 9, 10].

There have been extensive study and application of \mathbb{F}_q -linear \mathbb{F}_{q^t} -codes, where $t \geq 2$ is an integer. Cao et al. [4] studied the structure and a canonical form decomposition of any λ -constacyclic \mathbb{F}_q -linear code over \mathbb{F}_{q^t} . Furthermore, Cao et al. [5] gave the structure and canonical form decompositions of semisimple multivariable \mathbb{F}_q -linear codes over \mathbb{F}_{q^t} . Huffman [8] placed two different trace inner products on \mathbb{F}_q -linear \mathbb{F}_{q^t} -codes and examined the case $t = 2$ in detail giving specific bases and enumeration for the self-orthogonal and self-dual cyclic codes. This paper is a generalization of Huffman [8]. 0-trace bilinear form for all t and γ -trace bilinear form for t even on $\mathbb{F}_{q^t}^n$ have been studied in [8]. Sharma and Kaur placed a new trace bilinear form on $\mathbb{F}_{q^t}^n$ which is called the $*$ bilinear form for all t , but they did not consider the Δ -bilinear form for t even. In this paper, we give a new trace bilinear form on $\mathbb{F}_{q^t}^n$ which is called Δ -bilinear form for t even, where n is a positive integer coprime to q , and study the bases and enumeration of cyclic Δ -self-orthogonal and cyclic Δ -self-dual \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes. Our theory and method could be employed to obtain many good codes, which give the same parameters as the best known linear codes.

The present paper is organized as follows. In Section 2, we sketch the basic Lemmas needed in this paper. Section 3 gives a new trace bilinear form on $\mathbb{F}_{q^t}^n$ which is called Δ -bilinear form, where n is a positive integer coprime to q . According to this new trace bilinear form, bases and enumeration of cyclic Δ -self-orthogonal and cyclic Δ -self-dual \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes are investigated, respectively, in Section 4. Finally, we describe a program to construct cyclic Δ -self-orthogonal and cyclic Δ -self-dual \mathbb{F}_3 -linear \mathbb{F}_{3^2} -codes of length 7 and construct some good \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes in Section 5.

2 Preliminaries

Let $\mathcal{R}_n^{(q)}$ and $\mathcal{R}_n^{(q^t)}$ denote the group algebra $\mathbb{F}_q[X]/\langle X^n - 1 \rangle$ and $\mathbb{F}_{q^t}[X]/\langle X^n - 1 \rangle$, respectively, where X is an indeterminate over \mathbb{F}_p and any extension field of \mathbb{F}_p , n is a positive integer coprime to q and $t \geq 2$ is an integer. As $\gcd(n, q) = 1$, $X^n - 1$ has distinct roots and $\mathcal{R}_n^{(q)}$, $\mathcal{R}_n^{(q^t)}$ are semi-simple. Furthermore, $\mathcal{R}_n^{(q)}$ and $\mathcal{R}_n^{(q^t)}$ can be written as a direct sum of its minimal ideals, respectively, all of which are fields. A minimal ideal is one which does not contain any smaller nonzero ideal. If \mathcal{C} is a linear code over \mathbb{F}_{q^t} with parameters $[n, k, d]$, where n is the length of \mathcal{C} , k is the dimension of \mathcal{C} and d is the minimum Hamming distance of \mathcal{C} , then \mathcal{C} is called maximum distance separable, or MDS for short, if $d = n - k + 1$. Throughout this paper, $\dim_K V$ denotes the dimension of a finite-dimensional vector space V over the field K .

Let $X^n - 1 = m_0(X)m_1(X)\cdots m_{s-1}(X)$, where $m_i(X)$ is a monic irreducible polynomial over \mathbb{F}_q for $0 \leq i \leq s-1$ and $m_0(X) = -1 + X$. Let η' be a fixed primitive n th root of unity over the splitting field of $X^n - 1$ over

\mathbb{F}_p . Then $C_{l_i}^{(q)} = \{l_i, l_i q, l_i q^2, \dots\} \pmod{n}$ is the q -cyclotomic coset modulo n and l_i is chosen so that $\{\eta^{l_i k} | k \in C_{l_i}^{(q)}\}$ are the roots of $m_i(X)$ in a splitting field of $X^n - 1$ over \mathbb{F}_q , where $0 \leq i \leq s - 1$.

Next, we sum up some results from [8] and restate it as Lemmas 1 and 2.

Lemma 1 [8, Lemma 1] *For any integer l , $C_l^{(q)} = C_l^{(q^t)} \cup C_{lq}^{(q^t)} \cup \dots \cup C_{lq^{a-1}}^{(q^t)}$, where the union is a disjoint union and $a = \gcd(t, |C_l^{(q)}|)$. Furthermore, for $0 \leq i \leq a - 1$,*

$$|C_{lq^i}^{(q^t)}| = |C_l^{(q^t)}| = \frac{|C_l^{(q)}|}{\gcd(t, |C_l^{(q)}|)}.$$

Lemma 2 [8, Theorem 1] *The following hold.*

(i) $X^n - 1 = m_0(X)m_1(X) \cdots m_{s-1}(X)$, where $m_i(X)$ is a monic irreducible polynomial over \mathbb{F}_q for $0 \leq i \leq s - 1$ with $m_i(X) \leftrightarrow C_{l_i}^{(q)}$ and $m_0(X) = -1 + X \leftrightarrow C_0^{(q)} = \{0\}$ and \leftrightarrow gives the association between $m_i(X)$ and cyclotomic coset.

(ii) For $0 \leq i \leq s - 1$, $m_i(X) = M_{i,0}(X)M_{i,1}(X) \cdots M_{i,s_i-1}(X)$, where $M_{i,j}(X)$ is a monic irreducible polynomial over \mathbb{F}_{q^t} with $M_{i,j}(X) \leftrightarrow C_{l_i q^j}^{(q^t)}$ for $0 \leq j \leq s_i - 1$. Also $m_0(X) = -1 + X = M_{0,0}(X) \leftrightarrow C_0^{(q^t)} = \{0\}$ with $s_0 = 1$. Furthermore, the factorization of $X^n - 1$ into monic irreducible polynomials over \mathbb{F}_{q^t} is given by $X^n - 1 = \prod_{i=0}^{s-1} \prod_{j=0}^{s_i-1} M_{i,j}(X)$.

(iii) For $0 \leq i \leq s - 1$, $\deg m_i(X) = |C_{l_i}^{(q)}|$. In addition, $s_i = \gcd(t, |C_{l_i}^{(q)}|)$ and $\deg M_{i,j}(X) = |C_{l_i q^j}^{(q^t)}| = |C_{l_i}^{(q)}| / \gcd(t, |C_{l_i}^{(q)}|)$ for $0 \leq j \leq s_i - 1$.

(iv) $\mathcal{R}_n^{(q)} = \mathcal{K}_0 \oplus \mathcal{K}_1 \oplus \cdots \oplus \mathcal{K}_{s-1}$, where $\mathcal{K}_i \leftrightarrow C_{l_i}^{(q)}$ and \mathcal{K}_i is the ideal of $\mathcal{R}_n^{(q)}$ generated by $\hat{m}_i(X) = (X^n - 1)/m_i(X)$. $\mathcal{K}_i \cong \mathbb{F}_{q^{d_i}}$, where $d_i = |C_{l_i}^{(q)}|$ and $\mathcal{K}_i \mathcal{K}_j = \{0\}$ if $i \neq j$.

(v) $\mathcal{R}_n^{(q^t)} = \mathcal{I}_{0,0} \oplus \mathcal{I}_{1,0} \oplus \cdots \oplus \mathcal{I}_{1,s_1-1} \oplus \cdots \oplus \mathcal{I}_{s-1,0} \oplus \cdots \oplus \mathcal{I}_{s-1,s_{s-1}-1}$, where $\mathcal{I}_{i,j} \leftrightarrow C_{l_i q^j}^{(q^t)}$ and $\mathcal{I}_{i,j}$ is the ideal of $\mathcal{R}_n^{(q^t)}$ generated by $\hat{M}_{i,j}(X) = (X^n - 1)/M_{i,j}(X)$. $\mathcal{I}_{i,j} \cong \mathbb{F}_{q^{tD_i}}$, where $D_i = |C_{l_i q^j}^{(q^t)}| = d_i/s_i$ for $0 \leq i \leq s - 1$, $0 \leq j \leq s_i - 1$ and $\mathcal{I}_{i,j} \mathcal{I}_{i',j'} = \{0\}$ if $(i, j) \neq (i', j')$.

The relationship between the minimal ideals \mathcal{K}_i and $\mathcal{I}_{i,j}$ determines the nature of the containment $\mathcal{R}_n^{(q)} \subset \mathcal{R}_n^{(q^t)}$. Huffman [8] defined the ring automorphism $\tau_{q^w, u} : \mathcal{R}_n^{(q^r)} \rightarrow \mathcal{R}_n^{(q^r)}$ as $\tau_{q^w, u}(\sum_{k=0}^{n-1} a_k X^k) = \sum_{k=0}^{n-1} a_k^{q^w} X^{uk}$, where w, r, u are integers with $0 \leq w \leq r$, $1 \leq u \leq n - 1$ and $\gcd(u, n) = 1$. When $r = t$, the map $\tau_{q^w, u} : \mathcal{R}_n^{(q^t)} \rightarrow \mathcal{R}_n^{(q^t)}$ is a ring automorphism and permutes the minimal ideals $\mathcal{I}_{i,j}$ given in Lemma 2 (v). This permutation action can be described completely by the cyclotomic cosets.

Lemma 3 [8, Lemma 2] For $0 \leq i \leq s-1$ and $0 \leq j \leq s_i-1$, we have $\tau_{q^w,u}(\mathcal{I}_{i,j}) = \mathcal{I}_{i',j'}$, where i' and j' are determined as follows. If $\mathcal{I}_{i,j} \leftrightarrow C_l^{(q^t)}$, then $\mathcal{I}_{i',j'} \leftrightarrow C_{lu^{-1}q^w}^{(q^t)}$.

In [8], we have $\mathcal{K}_i \subset \mathcal{J}_i = \mathcal{I}_{i,0} \oplus \mathcal{I}_{i,1} \oplus \cdots \oplus \mathcal{I}_{i,s_i-1}$ for $0 \leq i \leq s-1$. So we can determine the precise containment.

Lemma 4 [8, Theorem 2] For $0 \leq i \leq s-1$,

$$\begin{aligned} \mathcal{K}_i &= \{f(X) \in \mathcal{J}_i \mid \tau_{q,1}(f(X)) = f(X)\} \\ &= \{c(X) + \tau_{q,1}(c(X)) + \cdots + \tau_{q^{s_i-1},1}(c(X)) \mid c(X) \in \mathcal{I}_{i,0} \\ &\quad \text{and } \tau_{q^{s_i},1}(c(X)) = c(X)\}. \end{aligned}$$

Now, we define the trace map $Tr_{Q,r} : \mathbb{F}_{Q^r} \rightarrow \mathbb{F}_Q$ by $Tr_{Q,r}(b) = \sum_{w=0}^{r-1} b^{Q^w}$ for $b \in \mathbb{F}_{Q^r}$, where Q is a power of q and r is a positive integer. Let $t = 2^a m$, $A = 2^{a-1}$ and $Q = q^A$, where $a \geq 1$ and m is odd. Then $t = 2Am$. Since $2A \mid t$, $\mathbb{F}_{Q^2} = \mathbb{F}_{q^{2A}}$ is a subfield of \mathbb{F}_{q^t} . By [8], there exists an element $0 \neq \gamma \in \mathbb{F}_{Q^2} \subseteq \mathbb{F}_{q^t}$ such that $Tr_{Q,2}(\gamma) = \gamma + \gamma^Q = 0$, so $\gamma^Q = -\gamma$. When $t = 2$, we have $\gamma^q = -\gamma$. Huffman [8] determined the Hermitian trace inner product as follows.

$$\langle a, b \rangle_\gamma = \sum_{i=0}^{n-1} Tr_{q,t}(\gamma a_i b_i^{q^{t/2}}) = \sum_{i=0}^{n-1} \sum_{w=0}^{t-1} (\gamma a_i b_i^{q^{t/2}})^{q^w},$$

where $a = (a_0, a_1, \dots, a_{n-1})$, $b = (b_0, b_1, \dots, b_{n-1}) \in \mathbb{F}_{q^t}^n$. Then, the Hermitian trace bilinear form on $\mathcal{R}_n^{(q^t)}$ was determined as bellow.

$$(a(X), b(X))_\gamma = \sum_{w=0}^{t-1} \tau_{q^w,1}(\gamma a(X) \tau_{q^{t/2},-1}(b(X))),$$

where $a(X) = a_0 + a_1 X + \cdots + a_{n-1} X^{n-1}$, $b(X) = b_0 + b_1 X + \cdots + b_{n-1} X^{n-1} \in \mathcal{R}_n^{(q^t)}$.

If $a = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{F}_{q^t}^n$, define $\sigma(a) = (a_{n-1}, a_0, a_1, \dots, a_{n-2})$ is the cyclic shift of a . A permutation μ of $\{0, 1, 2, \dots, s-1\}$ is defined by $\tau_{1,-1}(\mathcal{J}_i) = \tau_{q^{t/2},-1}(\mathcal{J}_i) = \mathcal{J}_{\mu(i)}$ for $0 \leq i \leq s-1$. At the end of this section, we list several basic Lemmas 5-7 needed in the following sections.

Lemma 5 [8, Theorem 7] Let $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{s-1}$ and $\mathcal{C}^{\perp_\gamma} = \mathcal{C}'_0 \oplus \mathcal{C}'_1 \oplus \cdots \oplus \mathcal{C}'_{s-1}$, where $\mathcal{C}^{\perp_\gamma} = \{v \in \mathbb{F}_{q^t}^n \mid \langle c, v \rangle_\gamma = 0 \text{ for all } c \in \mathcal{C}\}$, $\mathcal{C}_i = \mathcal{C} \cap \mathcal{J}_i$ and $\mathcal{C}'_i = \mathcal{C}^{\perp_\gamma} \cap \mathcal{J}_i$ for all $0 \leq i \leq s-1$. Then, for each i , we have

$$\mathcal{C}'_{\mu(i)} = \{a(X) \in \mathcal{J}_{\mu(i)} \mid (c(X), a(X))_\gamma = 0 \text{ for all } c(X) \in \mathcal{C}_i\}.$$

Furthermore, if the \mathcal{K}_i -dimension of \mathcal{C}_i is k_i , then the $\mathcal{K}_{\mu(i)}$ -dimension of $\mathcal{C}'_{\mu(i)}$ is $t - k_i$.

Lemma 6 [8, Lemma 10] Suppose that $\mu(i) = i$, where $1 \leq i \leq s-1$. The following hold.

- (i) d_i is even unless n is even and $l_i = n/2$.
- (ii) If n is even, for some $1 \leq i^\# \leq s-1$, then $l_{i^\#} = n/2$, $C_{l_{i^\#}}^{(q)} = C_{l_{i^\#}}^{(q^t)} = \{n/2\}$, $d_{i^\#} = s_{i^\#} = 1$ and $\mu(i^\#) = i^\#$.
- (iii) If $i \neq i^\#$ and s_i is odd, then $\tau_{1,-1}(\mathcal{I}_{i,j}) = \mathcal{I}_{i,j}$ for all $0 \leq j < s_i$.
- (iv) If $i \neq i^\#$ and s_i is even, then either $\tau_{1,-1}(\mathcal{I}_{i,j}) = \mathcal{I}_{i,j}$ for all $0 \leq j < s_i$ or $\tau_{1,-1}(\mathcal{I}_{i,j}) = \mathcal{I}_{i,j+s_i/2}$ for all $0 \leq j < s_i$, where the second subscript is computed modulo s_i .
- (v) If $i \neq i^\#$ and $\tau_{1,-1}(\mathcal{I}_{i,0}) = \mathcal{I}_{i,0}$, then D_i is even and $-l_i \equiv l_i q^{D_i/2} \pmod{n}$.

Lemma 7 [8, Lemma 11] If n is even, let $i^\#$ be chosen so that $l_{i^\#} = n/2$. The following hold.

- (i) $\tau_{1,-1}$ is the identity map on \mathcal{J}_0 and $\tau_{q,1}(a(X)) = a(X)^q$ for all $a(X) \in \mathcal{J}_0$.
- (ii) If n is even, then $\tau_{1,-1}$ is the identity map on $\mathcal{J}_{i^\#} = \mathcal{I}_{i^\#,0}$ and $\tau_{q,1}(a(X)) = a(X)^q$ for all $a(X) \in \mathcal{J}_{i^\#}$.
- (iii) Suppose $i \neq 0$ and $i \neq i^\#$. If $\tau_{1,-1}(\mathcal{I}_{i,j}) = \mathcal{I}_{i,j}$, then $\mu(i) = i$ and $\tau_{1,-1}(a(X)) = a(X)^{q^{tD_i/2}}$ for all $a(X) \in \mathcal{I}_{i,j}$.
- (iv) Suppose $i \neq 0$, $i \neq i^\#$ and t is even. If $\tau_{q^{t/2},-1}(\mathcal{I}_{i,j}) = \mathcal{I}_{i,j}$, then $\mu(i) = i$ and $\tau_{q^{t/2},-1}(a(X)) = a(X)^{q^{tD_i/2}}$ for all $a(X) \in \mathcal{I}_{i,j}$.
- (v) If $i \neq 0$, $i \neq i^\#$ and t is even, there does not exist j with $0 \leq j < s_i$ such that both $\tau_{1,-1}(\mathcal{I}_{i,j}) = \mathcal{I}_{i,j}$ and $\tau_{q^{t/2},-1}(\mathcal{I}_{i,j}) = \mathcal{I}_{i,j}$ hold.

3 Δ -Trace bilinear form on $\mathbb{F}_{q^t}^n$ and $\mathcal{R}_n^{(q^t)}$

In this section, we define a new trace bilinear form on $\mathbb{F}_{q^t}^n$ and $\mathcal{R}_n^{(q^t)}$ for any even integer $t \geq 2$ satisfying $t \not\equiv 1 \pmod{p}$ and study its properties. We need the following Lemma.

Lemma 8 Let $t = 2^a m \geq 2$ be an even integer satisfying $t \not\equiv 1 \pmod{p}$, where $a \geq 1$ and m is odd. Then there exists an element $\gamma \in \mathbb{F}_{q^{2a}} \subseteq \mathbb{F}_{q^t}$ with $\gamma \neq 0$ such that $\gamma + \gamma^{q^{2^{a-1}}} = 0$. Thus the map $\psi : \mathbb{F}_{q^t} \rightarrow \mathbb{F}_{q^t}$ defined as $\psi(\alpha) = \alpha^q + \alpha^{q^2} + \cdots + \alpha^{q^{t-1}} = \text{Tr}_{q,t}(\alpha) - \alpha$ is an \mathbb{F}_q -linear vector space automorphism, where $\alpha \in \mathbb{F}_{q^t}$.

Proof The result will be proved if we can show ψ is an injective \mathbb{F}_q -linear vector space automorphism. It can easily be verified that ψ is an \mathbb{F}_q -linear map. It remains to show that the kernel of ψ denoted by $\text{Ker}(\psi)$ is $\{0\}$. If $\alpha \in \text{Ker}(\psi)$, then $\psi(\alpha) = \alpha^q + \alpha^{q^2} + \cdots + \alpha^{q^{t-1}} = 0$. Furthermore, we have $\psi(\alpha)^q = \alpha + \alpha^{q^2} + \cdots + \alpha^{q^{t-1}} = 0$. Then, we obtain $\psi(\alpha) - \psi(\alpha)^q = \alpha^q - \alpha = 0$, which implies $\alpha^{q^u} = \alpha$ for each integer u ($1 \leq u \leq t-1$). This implies that $(t-1)\alpha = \psi(\alpha) = 0$. As $t \not\equiv 1 \pmod{p}$, we have $(t-1)\alpha = 0$ if and only if $\alpha = 0$. The proof is completed.

Now, we define a trace inner product $(\cdot, \cdot)_\Delta : \mathbb{F}_{q^t}^n \times \mathbb{F}_{q^t}^n \rightarrow \mathbb{F}_q$ as

$$(a, b)_\Delta = \sum_{j=0}^{n-1} \text{Tr}_{q,t}(\gamma a_j \psi(b_j^{q^{t/2}}))$$

for all $a = (a_0, a_1, \dots, a_{n-1})$, $b = (b_0, b_1, \dots, b_{n-1})$ in $\mathbb{F}_{q^t}^n$. In the following Lemma, we prove that the trace inner product $(\cdot, \cdot)_\Delta$ is a non-degenerate trace bilinear form on $\mathbb{F}_{q^t}^n$ for any prime power q .

Lemma 9 *For $a, b, c \in \mathbb{F}_{q^t}^n$ and $\alpha \in \mathbb{F}_q$, the following hold.*

- (i) $(a, b)_\Delta \in \mathbb{F}_q$.
- (ii) $(a, b+c)_\Delta = (a, b)_\Delta + (a, c)_\Delta$ and $(a+b, c)_\Delta = (a, c)_\Delta + (b, c)_\Delta$.
- (iii) $(\alpha a, b)_\Delta = (a, \alpha b)_\Delta = \alpha(a, b)_\Delta$.
- (iv) $(\cdot, \cdot)_\Delta$ is non-degenerate.

Proof To prove these results, we let $a = (a_0, a_1, \dots, a_{n-1})$, $b = (b_0, b_1, \dots, b_{n-1})$ and $c = (c_0, c_1, \dots, c_{n-1})$, where $a_i, b_i, c_i \in \mathbb{F}_{q^t}$ for all $0 \leq i \leq n-1$.

(i) As $(a, b)_\Delta = \sum_{i=0}^{n-1} \text{Tr}_{q,t}(\gamma a_i \psi(b_i^{q^{t/2}}))$ and $\text{Tr}_{q,t}(\gamma a_i \psi(b_i^{q^{t/2}})) \in \mathbb{F}_q$ for $0 \leq i \leq n-1$, (i) holds.

Parts (ii) and (iii) follow because $\text{Tr}_{q,t}$ is \mathbb{F}_q -linear.

(iv) To prove this, we need to show that if $(a, b)_\Delta = 0$ for all $b \in \mathbb{F}_{q^t}^n$, then $a = 0$. Suppose, on the contrary, that there exists a non-zero $a = (a_0, a_1, \dots, a_{n-1}) \in \mathbb{F}_{q^t}^n$ such that $(a, b)_\Delta = 0$ for all $b \in \mathbb{F}_{q^t}^n$. In particular, let $a_j \neq 0$ for some j . As $\text{Tr}_{q,t}$ is an onto map and $\gamma \neq 0$, there exists $d \in \mathbb{F}_{q^t}$ such that $\text{Tr}_{q,t}(\gamma a_j \psi(d^{q^{t/2}})) \neq 0$. Let $b = (b_0, b_1, \dots, b_{n-1}) \in \mathbb{F}_{q^t}^n$, where $b_i = 0$ for all $i \neq j$ and $b_j = d$. Then, $\text{Tr}_{q,t}(\gamma a_j \psi(b_j^{q^{t/2}})) \neq 0$. This leads to a contradiction.

Next, we define a form $[\cdot, \cdot]_\Delta : \mathcal{R}_n^{(q^t)} \times \mathcal{R}_n^{(q^t)} \rightarrow \mathcal{R}_n^{(q)}$ as follows.

$$[a(X), b(X)]_\Delta = \sum_{u=0}^{t-1} \tau_{q^u, 1}(\gamma a(X) \sum_{w=1}^{t-1} \tau_{q^{t/2+w}, -1}(b(X)))$$

for all $a(X), b(X) \in \mathcal{R}_n^{(q^t)}$. We prove a result for this form analogous to Lemma 9 as follows.

Lemma 10 *Let $a(X), b(X), c(X) \in \mathcal{R}_n^{(q^t)}$. The following hold.*

- (i) $[a(X), b(X)]_\Delta = \sum_{k=0}^{n-1} (a, \sigma^k(b))_\Delta X^k$.
- (ii) $[a(X), b(X)]_\Delta \in \mathcal{R}_n^{(q)}$.
- (iii) $[a(X), b(X) + c(X)]_\Delta = [a(X), b(X)]_\Delta + [a(X), c(X)]_\Delta$ and $[a(X) + b(X), c(X)]_\Delta = [a(X), c(X)]_\Delta + [b(X), c(X)]_\Delta$.

(iv) For $f(X) \in \mathcal{R}_n^{(q)}$, we have $[f(X)a(X), b(X)]_\Delta = f(X)[a(X), b(X)]_\Delta$ and

$$[a(X), f(X)b(X)]_\Delta = \tau_{1,-1}(f(X))[a(X), b(X)]_\Delta.$$

(v) $[\cdot, \cdot]_\Delta$ is non-degenerate.

Proof (i) According to the definition of $[\cdot, \cdot]_\Delta$, it follows that $[a(X), b(X)]_\Delta = \sum_{u=0}^{t-1} \tau_{q^u, 1}(\gamma a(X) \sum_{w=1}^{t-1} \tau_{q^{t/2+w}, -1}(b(X)))$. For $\gamma a(X) \sum_{w=1}^{t-1} \tau_{q^{t/2+w}, -1}(b(X))$, we have the following results.

(1) When $w = 1$, we have

$$\begin{aligned} & \gamma a(X) \tau_{q^{t/2+1}, -1}(b(X)) \\ &= \gamma(a_0 + a_1 X + a_2 X^2 + a_3 X^3 + \cdots + a_{n-2} X^{n-2} + a_{n-1} X^{n-1}) \\ & \quad \cdot (b_0^{q^{t/2+1}} + b_{n-1}^{q^{t/2+1}} X + b_{n-2}^{q^{t/2+1}} X^2 + \cdots + b_2^{q^{t/2+1}} X^{n-2} + b_1^{q^{t/2+1}} X^{n-1}) \\ &= c_{0,1} + c_{1,1} X + c_{2,1} X^2 + \cdots + c_{n-2,1} X^{n-2} + c_{n-1,1} X^{n-1}, \end{aligned}$$

where

$$\begin{aligned} c_{0,1} &= \gamma(a_0 b_0^{q^{t/2+1}} + a_1 b_1^{q^{t/2+1}} + a_2 b_2^{q^{t/2+1}} + \cdots + a_{n-2} b_{n-2}^{q^{t/2+1}} + a_{n-1} b_{n-1}^{q^{t/2+1}}); \\ c_{1,1} &= \gamma(a_0 b_{n-1}^{q^{t/2+1}} + a_1 b_0^{q^{t/2+1}} + a_2 b_1^{q^{t/2+1}} + \cdots + a_{n-2} b_{n-3}^{q^{t/2+1}} + a_{n-1} b_{n-2}^{q^{t/2+1}}); \\ c_{2,1} &= \gamma(a_0 b_{n-2}^{q^{t/2+1}} + a_1 b_{n-1}^{q^{t/2+1}} + a_2 b_0^{q^{t/2+1}} + \cdots + a_{n-2} b_{n-4}^{q^{t/2+1}} + a_{n-1} b_{n-3}^{q^{t/2+1}}); \\ & \quad \dots \\ c_{n-2,1} &= \gamma(a_0 b_2^{q^{t/2+1}} + a_1 b_3^{q^{t/2+1}} + a_2 b_4^{q^{t/2+1}} + \cdots + a_{n-2} b_0^{q^{t/2+1}} + a_{n-1} b_1^{q^{t/2+1}}); \\ c_{n-1,1} &= \gamma(a_0 b_1^{q^{t/2+1}} + a_1 b_2^{q^{t/2+1}} + a_2 b_3^{q^{t/2+1}} + \cdots + a_{n-2} b_{n-1}^{q^{t/2+1}} + a_{n-1} b_0^{q^{t/2+1}}). \end{aligned}$$

(2) When $w = 2$, we have

$$\begin{aligned} & \gamma a(X) \tau_{q^{t/2+2}, -1}(b(X)) \\ &= \gamma(a_0 + a_1 X + a_2 X^2 + a_3 X^3 + \cdots + a_{n-2} X^{n-2} + a_{n-1} X^{n-1}) \\ & \quad \cdot (b_0^{q^{t/2+2}} + b_{n-1}^{q^{t/2+2}} X + b_{n-2}^{q^{t/2+2}} X^2 + \cdots + b_2^{q^{t/2+2}} X^{n-2} + b_1^{q^{t/2+2}} X^{n-1}) \\ &= c_{0,2} + c_{1,2} X + c_{2,2} X^2 + \cdots + c_{n-2,2} X^{n-2} + c_{n-1,2} X^{n-1}, \end{aligned}$$

where

$$\begin{aligned} c_{0,2} &= \gamma(a_0 b_0^{q^{t/2+2}} + a_1 b_1^{q^{t/2+2}} + a_2 b_2^{q^{t/2+2}} + \cdots + a_{n-2} b_{n-2}^{q^{t/2+2}} + a_{n-1} b_{n-1}^{q^{t/2+2}}); \\ c_{1,2} &= \gamma(a_0 b_{n-1}^{q^{t/2+2}} + a_1 b_0^{q^{t/2+2}} + a_2 b_1^{q^{t/2+2}} + \cdots + a_{n-2} b_{n-3}^{q^{t/2+2}} + a_{n-1} b_{n-2}^{q^{t/2+2}}); \\ c_{2,2} &= \gamma(a_0 b_{n-2}^{q^{t/2+2}} + a_1 b_{n-1}^{q^{t/2+2}} + a_2 b_0^{q^{t/2+2}} + \cdots + a_{n-2} b_{n-4}^{q^{t/2+2}} + a_{n-1} b_{n-3}^{q^{t/2+2}}); \\ & \quad \dots \end{aligned}$$

$$c_{n-2,2} = \gamma(a_0 b_2^{q^{t/2+2}} + a_1 b_3^{q^{t/2+2}} + a_2 b_4^{q^{t/2+2}} + \cdots + a_{n-2} b_0^{q^{t/2+2}} + a_{n-1} b_1^{q^{t/2+2}});$$

$$c_{n-1,2} = \gamma(a_0 b_1^{q^{t/2+2}} + a_1 b_2^{q^{t/2+2}} + a_2 b_3^{q^{t/2+2}} + \cdots + a_{n-2} b_{n-1}^{q^{t/2+2}} + a_{n-1} b_0^{q^{t/2+2}}).$$

The rest, $3 \leq w \leq t-1$, can be done in the same manner. Therefore, we have

$$\gamma a(X) \sum_{w=1}^{t-1} \tau_{q^{t/2+w}, -1}(b(X)) = \sum_{k=0}^{n-1} \left(\sum_{\substack{i,j=0 \\ i-j \equiv k \pmod{n}}}^{n-1} \gamma a_i \psi(b_j^{q^{t/2}}) X^k \right).$$

Then, we have

$$\begin{aligned} [a(X), b(X)]_{\Delta} &= \sum_{u=0}^{t-1} \tau_{q^u, 1} \left(\sum_{k=0}^{n-1} \left(\sum_{\substack{i,j=0 \\ i-j \equiv k \pmod{n}}}^{n-1} \gamma a_i \psi(b_j^{q^{t/2}}) X^k \right) \right) \\ &= (a, b)_{\Delta} + (a, \sigma(b))_{\Delta} X + \cdots + (a, \sigma^{(n-1)}(b))_{\Delta} X^{n-1}, \end{aligned}$$

where the cyclic shift $\sigma(b) = (b_{n-1}, b_0, b_1, \dots, b_{n-2})$ of $b = (b_0, b_1, \dots, b_{n-1}) \in \mathbb{F}_{q^t}^n$ is identified with $Xb(X) \in \mathcal{R}_n^{(q^t)}$, (i) holds.

(ii) It follows from part (i) and Lemma 9 (i).

(iii) As $\tau_{q^u, 1}$ and $\tau_{q^u, -1}$ are ring automorphisms for any integer $u \geq 0$, part (iii) follows immediately.

(iv) For $f(X) \in \mathcal{R}_n^{(q)}$, as $\tau_{q,1}(f(X)) = f(X)$, we have $\tau_{q^{t/2}, -1}(f(X)) = \tau_{1, -1}(f(X))$. From this, (iv) holds.

(v) Part (v) will be proved by showing that if $[a(X), b(X)]_{\Delta} = 0$ for all $b(X) \in \mathcal{R}_n^{(q^t)}$, then $a(X) = 0$. If not, there would exist j ($0 \leq j \leq n-1$) such that $a_j \neq 0$. Since $Tr_{q,t}$ is an onto map and $\gamma \neq 0$, there exists $\theta \in \mathbb{F}_{q^t}$ such that $Tr_{q,t}(\gamma\theta) \neq 0$. Then for $b(X) = \psi^{-1}(\theta a_j^{-1} X^{-j})$, let $a = (a_0, a_1, \dots, a_{n-1})$ where $a_i = 0$ for all $i \neq j$ and $a_j \neq 0$. By part (i), we have $[a(X), b(X)]_{\Delta} = Tr_{q,t}(\gamma\theta) \neq 0$, which is a contradiction.

Now we proceed to study the dual codes of cyclic \mathbb{F}_q -linear \mathbb{F}_{q^t} -code \mathcal{C} of length n with respect to this new trace bilinear form $(\cdot, \cdot)_{\Delta}$ on $\mathbb{F}_{q^t}^n$.

4 Dual codes of cyclic \mathbb{F}_q -linear \mathbb{F}_{q^t} -codes

Let \mathcal{C} be a cyclic \mathbb{F}_q -linear \mathbb{F}_{q^t} -code of length n , where $\gcd(n, q) = 1$. Then, the Δ -dual code of \mathcal{C} is defined as $\mathcal{C}^{\perp_{\Delta}} = \{v \in \mathbb{F}_{q^t}^n \mid (c, v)_{\Delta} = 0 \text{ for all } c \in \mathcal{C}\}$. It is easy to verify that the dual code $\mathcal{C}^{\perp_{\Delta}}$ is also an \mathbb{F}_q -linear \mathbb{F}_{q^t} -code of length n . Furthermore, if \mathcal{C} is cyclic, then its dual code $\mathcal{C}^{\perp_{\Delta}}$ is also cyclic. From now onwards, throughout this paper, we will view the cyclic \mathbb{F}_q -linear \mathbb{F}_{q^t} -code \mathcal{C} of length n and its Δ -dual code $\mathcal{C}^{\perp_{\Delta}}$ as $\mathcal{R}_n^{(q)}$ -submodule of $\mathcal{R}_n^{(q^t)}$, respectively. In addition, if $\mathcal{C} \subseteq \mathcal{R}_n^{(q^t)}$ is any cyclic \mathbb{F}_q -linear \mathbb{F}_{q^t} -code, then one can easily obtain its dual code $\mathcal{C}^{\perp_{\Delta}} \subseteq \mathcal{R}_n^{(q^t)}$ with respect to trace bilinear form $[\cdot, \cdot]_{\Delta}$.

Next, we study the properties of the Δ -dual codes of cyclic \mathbb{F}_q -linear \mathbb{F}_{q^t} -codes of length n . For the ring automorphism $\tau_{q^u, -1}$ ($0 \leq u \leq t-1$) on the ideal \mathcal{J}_i ($0 \leq i \leq s-1$) of $\mathcal{R}_n^{(q^t)}$, we observe that $\mathcal{C}_{-l_0}^{(q)} = \mathcal{C}_{l_0}^{(q)}$, and further for each i ($1 \leq i \leq s-1$), there exists a unique integer i' ($1 \leq i' \leq s-1$) satisfying $\mathcal{C}_{-l_i}^{(q)} = \mathcal{C}_{l_{i'}}^{(q)}$. This gives rise to a permutation μ of $\{0, 1, 2, \dots, s-1\}$ defined by $\mathcal{C}_{-l_i}^{(q)} = \mathcal{C}_{l_{\mu(i)}}^{(q)}$ for $0 \leq i \leq s-1$. It is not difficult to show that $\mu(0) = 0$ and $\mu(\mu(i)) = i$ for $0 \leq i \leq s-1$. That is, μ is either the identity permutation or a product of transpositions. When n is even, by Lemma 6, there exists an integer $i^\#$ ($1 \leq i^\# \leq s-1$) satisfying $\mathcal{C}_{l_{i^\#}}^{(q)} = \mathcal{C}_{\frac{n}{2}}^{(q)} = \{\frac{n}{2}\}$, as q is odd. Note that $\mathcal{C}_{-l_{i^\#}}^{(q)} = \mathcal{C}_{l_{i^\#}}^{(q)}$, so $\mu(i^\#) = i^\#$. With the help of the above concepts, we give the following Lemma.

Lemma 11 *Let u ($0 \leq u \leq t-1$) be a fixed integer. Then we have $\tau_{q^u, -1}(\mathcal{J}_i) = \mathcal{J}_{\mu(i)}$ for $0 \leq i \leq s-1$.*

Proof Working in a similar way as in Lemma 8 of Huffman [8], the result follows.

We are interested in the relationship between \mathcal{C} and $\mathcal{C}^{\perp_\Delta}$ viewed as $\mathcal{R}_n^{(q^t)}$ -submodules of $\mathcal{R}_n^{(q^t)}$. The following result will prove quite useful.

Lemma 12 *Let $a_i(X), b_i(X) \in \mathcal{J}_i$ for $0 \leq i \leq s-1$. If $a(X) = \sum_{i=0}^{s-1} a_i(X)$ and $b(X) = \sum_{i=0}^{s-1} b_i(X)$, then $[a(X), b(X)]_\Delta = \sum_{i=0}^{s-1} [a_i(X), b_{\mu(i)}(X)]_\Delta$.*

Proof By Lemma 10 (iii), we have $[a(X), b(X)]_\Delta = \sum_{i=0}^{s-1} \sum_{j=0}^{s-1} [a_i(X), b_j(X)]_\Delta$. It suffices to prove the result in the case that $[a_i(X), b_j(X)]_\Delta = 0$ if $j \neq \mu(i)$.

By Lemma 11, we have $\sum_{w=1}^{t-1} \tau_{q^{t/2+w}, -1}(b_j(X)) \in \mathcal{J}_{\mu(j)}$. If $j \neq \mu(i)$, then we have $\mu(j) \neq \mu(\mu(i)) = i$. By Lemma 2, we have that $\mathcal{J}_i \mathcal{J}_{\mu(j)} = \{0\}$ implying $\gamma a_i(X) \sum_{w=1}^{t-1} \tau_{q^{t/2+w}, -1}(b_j(X)) = 0$. By the definition of $[\cdot, \cdot]_\Delta$, this shows that $[a_i(X), b_j(X)]_\Delta = 0$ if $j \neq \mu(i)$.

Corollary 1 *Let $a, b \in \mathbb{F}_{q^t}^n$ be identified with $a(X) = \sum_{i=0}^{s-1} a_i(X)$ and $b(X) = \sum_{i=0}^{s-1} b_i(X)$, respectively, where $a_i(X), b_i(X) \in \mathcal{J}_i$ for $0 \leq i \leq s-1$. Then $b \in \mathcal{C}^{\perp_\Delta}$ if and only if $[a_i(X), b_{\mu(i)}(X)]_\Delta = 0$ for all $0 \leq i \leq s-1$ and all $a \in \mathcal{C}$.*

Proof Since \mathcal{C} is cyclic, $b \in \mathcal{C}^{\perp_\Delta}$ if and only if $(a, b)_\Delta = 0$ for all $a \in \mathcal{C}$ if and only if $(a, \sigma^k(b))_\Delta = 0$ for all $a \in \mathcal{C}$ and any integer k if and only if $[a(X), b(X)]_\Delta = 0$ for all $a(X) \in \mathcal{C}$ by Lemma 10 (i). By Lemma 12, we have $[a(X), b(X)]_\Delta = \sum_{i=0}^{s-1} [a_i(X), b_{\mu(i)}(X)]_\Delta$. For $0 \leq i \leq s-1$, by definitions of μ and $[\cdot, \cdot]_\Delta$, we have $[a_i(X), b_{\mu(i)}(X)]_\Delta \in \mathcal{J}_i$. Therefore, as $\mathcal{R}_n^{(q^t)} = \mathcal{J}_0 \oplus \mathcal{J}_1 \oplus \dots \oplus \mathcal{J}_{s-1}$ is a direct sum, $[a(X), b(X)]_\Delta = \sum_{i=0}^{s-1} [a_i(X), b_{\mu(i)}(X)]_\Delta = 0$ if and only if $[a_i(X), b_{\mu(i)}(X)]_\Delta = 0$ for all $0 \leq i \leq s-1$. The proof is completed.

According to Lemma 5, we have the following Theorem.

Theorem 1 *Let \mathcal{C} be a cyclic \mathbb{F}_q -linear \mathbb{F}_{q^t} -code of length n and $\mathcal{C}^{\perp_\Delta}$ the dual code of \mathcal{C} . Then we have $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{s-1}$ and $\mathcal{C}^{\perp_\Delta} = \mathcal{C}_0^{(\Delta)} \oplus \mathcal{C}_1^{(\Delta)} \oplus \cdots \oplus \mathcal{C}_{s-1}^{(\Delta)}$, where $\mathcal{C}_i = \mathcal{C} \cap \mathcal{I}_i$ and $\mathcal{C}_i^{(\Delta)} = \mathcal{C}^{\perp_\Delta} \cap \mathcal{I}_i$ for all $0 \leq i \leq s-1$. Thus, for each i , we have*

$$\mathcal{C}_{\mu(i)}^{(\Delta)} = \{a(X) \in \mathcal{I}_{\mu(i)} \mid [c(X), a(X)]_\Delta = 0 \text{ for all } c(X) \in \mathcal{C}_i\}.$$

Furthermore, if the \mathcal{K}_i -dimension of \mathcal{C}_i is k_i , then the $\mathcal{K}_{\mu(i)}$ -dimension of $\mathcal{C}_{\mu(i)}^{(\Delta)}$ is $t - k_i$.

Proof The proof of this result is quite similar to Theorem 7 of [8] and so is omitted.

In addition, a cyclic \mathbb{F}_q -linear \mathbb{F}_{q^t} -code \mathcal{C} of length n is said to be cyclic Δ -self-orthogonal if it satisfies $\mathcal{C} \subseteq \mathcal{C}^{\perp_\Delta}$, and it is called cyclic Δ -self-dual if it satisfies $\mathcal{C} = \mathcal{C}^{\perp_\Delta}$. Cyclic Δ -self-orthogonal and cyclic Δ -self-dual \mathbb{F}_q -linear \mathbb{F}_{q^t} -codes are characterized in the following Lemma.

Lemma 13 *Let \mathcal{C} be a cyclic \mathbb{F}_q -linear \mathbb{F}_{q^t} -code of length n and $\mathcal{C}^{\perp_\Delta}$ the dual code of \mathcal{C} . Let us write $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{s-1}$ and $\mathcal{C}^{\perp_\Delta} = \mathcal{C}_0^{(\Delta)} \oplus \mathcal{C}_1^{(\Delta)} \oplus \cdots \oplus \mathcal{C}_{s-1}^{(\Delta)}$, where $\mathcal{C}_i = \mathcal{C} \cap \mathcal{I}_i$ and $\mathcal{C}_i^{(\Delta)} = \mathcal{C}^{\perp_\Delta} \cap \mathcal{I}_i$ for all $0 \leq i \leq s-1$. Then,*

- (i) \mathcal{C} is cyclic Δ -self-orthogonal if and only if $\mathcal{C}_i \subseteq \mathcal{C}_i^{(\Delta)}$ for all $0 \leq i \leq s-1$.
- (ii) \mathcal{C} is cyclic Δ -self-dual if and only if $\mathcal{C}_i = \mathcal{C}_i^{(\Delta)}$ for all $0 \leq i \leq s-1$.

Proof The proof is trivial.

Now we consider the case $t = 2$ and study cyclic Δ -self-orthogonal and cyclic Δ -self-dual \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes. It is not difficult to find that, when $t = 2$, the minimal ideal $\mathcal{I}_{i,j}$ of $\mathcal{R}_n^{(q^2)}$ is the finite field of order q^{2D_i} for $0 \leq i \leq s-1$ and $0 \leq j \leq s_i - 1$, where $D_i = \frac{d_i}{s_i}$ and $s_i = \gcd(2, d_i)$. For $0 \leq i \leq s-1$, we choose primitive elements $\rho_{i,0}(X), \rho_{i,1}(X), \dots, \rho_{i,s_i-1}(X)$ of $\mathcal{I}_{i,0}, \mathcal{I}_{i,1}, \dots, \mathcal{I}_{i,s_i-1}$, respectively, satisfying $\tau_{q^j,1}(\rho_{i,0}(X)) = \rho_{i,j}(X)$ and $e_{i,j}(X)$ the identity of $\mathcal{I}_{i,j}$ for all $0 \leq j \leq s_i - 1$.

4.1 Cyclic Δ -self-orthogonal \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes

The bases of all cyclic Δ -self-orthogonal \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes of length n are determined in the following Theorem, where $\gcd(n, q) = 1$.

Theorem 2 *Let $t = 2$, q be a power of the prime p and $\gcd(n, q) = 1$. Let \mathcal{C} be a cyclic \mathbb{F}_q -linear \mathbb{F}_{q^2} -code of length n and $\mathcal{C}^{\perp_\Delta}$ the dual code of \mathcal{C} . We write $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{s-1}$ and $\mathcal{C}^{\perp_\Delta} = \mathcal{C}_0^{(\Delta)} \oplus \mathcal{C}_1^{(\Delta)} \oplus \cdots \oplus \mathcal{C}_{s-1}^{(\Delta)}$, where $\mathcal{C}_i = \mathcal{C} \cap \mathcal{I}_i$ and $\mathcal{C}_i^{(\Delta)} = \mathcal{C}^{\perp_\Delta} \cap \mathcal{I}_i$ for all $0 \leq i \leq s-1$. Then \mathcal{C} is cyclic*

Δ -self-orthogonal if and only if for each i ($0 \leq i \leq s-1$), the following hold.

(i) When $i = 0$ or, if n is even, $i = i^\#$, then

(a) $\mathcal{C}_i = \{0\}$, or

(b) \mathcal{C}_i is a 1-dimensional \mathcal{K}_i -subspace of \mathcal{J}_i with basis $\{\rho_{i,0}(X)^k\}$, where $k = 0$ when q is even and $k = \frac{q+1}{2}$ when q is odd.

(ii) When $i \neq 0$, $i \neq i^\#$, $\mu(i) = i$ and $\tau_{1,-1}(\mathcal{I}_{i,0}) = \mathcal{I}_{i,0}$, then

(a) $\mathcal{C}_i = \{0\}$, or

(b) \mathcal{C}_i is 1-dimensional over \mathcal{K}_i with basis $\{e_{i,0}(X) + \rho_{i,1}(X)^k\}$, where $k = (q^{d_i/2} - 1)m$ for $0 \leq m \leq q^{d_i/2}$.

(iii) When $i \neq 0$, $i \neq i^\#$, $\mu(i) = i$ and $\tau_{1,-1}(\mathcal{I}_{i,0}) = \mathcal{I}_{i,1}$, then

(a) $\mathcal{C}_i = \{0\}$, or

(b) \mathcal{C}_i is 1-dimensional over \mathcal{K}_i with bases $\{e_{i,0}(X)\}$, $\{e_{i,1}(X)\}$ and $\{e_{i,0}(X) + \rho_{i,1}(X)^k\}$, where $k = (q^{d_i/2} + 1)m$ for $0 \leq m \leq q^{d_i/2} - 2$.

(iv) When $\mu(i) \neq i$ and d_i is odd,

(a) if $\mathcal{C}_i = \{0\}$, then $\mathcal{C}_{\mu(i)} \subseteq \mathcal{J}_{\mu(i)}$;

(b) if $\mathcal{C}_i = \mathcal{J}_i$, then $\mathcal{C}_{\mu(i)} = \{0\}$;

(c) if \mathcal{C}_i is 1-dimensional over \mathcal{K}_i with basis $\{\rho_{i,0}(X)^k\}$, where $0 \leq k \leq q^{d_i}$, then $\mathcal{C}_{\mu(i)} \subseteq \mathcal{C}_{\mu(i)}^{(\Delta)}$ with $\{\rho_{\mu(i),0}(X)^{k'}\}$ the basis of $\mathcal{C}_{\mu(i)}^{(\Delta)}$ over $\mathcal{K}_{\mu(i)}$, where $k' = k = 0$ or $k' = q^{d_i} + 1 - k$ for $1 \leq k \leq q^{d_i}$.

(v) When $\mu(i) \neq i$ and d_i is even,

(a) if $\mathcal{C}_i = \{0\}$, then $\mathcal{C}_{\mu(i)} \subseteq \mathcal{J}_{\mu(i)}$;

(b) if $\mathcal{C}_i = \mathcal{J}_i$, then $\mathcal{C}_{\mu(i)} = \{0\}$;

(c) if \mathcal{C}_i is 1-dimensional over \mathcal{K}_i with basis $\{a(X)\}$, then $\mathcal{C}_{\mu(i)} \subseteq \mathcal{C}_{\mu(i)}^{(\Delta)}$ with $\{a'(X)\}$ the basis of $\mathcal{C}_{\mu(i)}^{(\Delta)}$ over $\mathcal{K}_{\mu(i)}$, where $a'(X) = e_{\mu(i),0}(X) + \rho_{\mu(i),1}(X)^{k'}$ when $a(X) = e_{i,0}(X) + \rho_{i,1}(X)^k$ with $k' = k = 0$ or $k' = q^{d_i} - 1 - k$ for $1 \leq k \leq q^{d_i} - 2$.

Proof (i) Suppose that $\mu(i) = i$. Since $\mathcal{C}_i \subseteq \mathcal{C}_i^{(\Delta)} = \mathcal{C}_{\mu(i)}^{(\Delta)}$ and the sum of the \mathcal{K}_i -dimensions of \mathcal{C}_i and $\mathcal{C}_{\mu(i)}^{(\Delta)}$ is at most 2, by Lemma 5, either \mathcal{C}_i is $\{0\}$ or \mathcal{C}_i is 1-dimensional over \mathcal{K}_i and $\mathcal{C}_i = \mathcal{C}_i^{(\Delta)}$.

Assume that $i = 0$ or, if n is even, $i = i^\#$. In this case, by Lemma 2 (iv), (v) and Lemma 6 (ii), we have $d_i = s_i = D_i = 1$, $\mathcal{J}_i = \mathcal{I}_{i,0} \cong \mathbb{F}_{q^2}$ and $\mathcal{K}_i \cong \mathbb{F}_q$. By the proof of Theorem 8 (i) of Huffman [8], we have $\{\rho_{i,0}(X)^k\}$ are bases of the $q+1$ distinct 1-dimensional \mathcal{K}_i -subspaces of \mathcal{J}_i , where $0 \leq k \leq q$. It is simple to show that $\{\rho_{i,0}(X)^k\}$ are bases of \mathcal{C}_i , where $0 \leq k \leq q$. As $\mathcal{C}_i = \mathcal{C}_i^{(\Delta)}$, we have $[\rho_{i,0}(X)^k, \rho_{i,0}(X)^k]_\Delta = 0$. By Lemma 7 (i) and (ii), we have $\tau_{1,-1}(c(X)) = c(X)$ and $\tau_{q,1}(c(X)) = c(X)^q$ for $c(X) \in \mathcal{J}_i$. Then, we have $\tau_{q,-1}(c(X)) = c(X)^q$ for $c(X) \in \mathcal{J}_i$. We will use the observation that $\tau_{q,1}(\gamma) = \gamma^q = -\gamma$ throughout the proof.

It is straightforward to show that (a) holds. For (b), we have

$$\begin{aligned} [a(X), b(X)]_\Delta &= \tau_{1,1}(\gamma a(X) \tau_{q^2,-1}(b(X))) + \tau_{q,1}(\gamma a(X) \tau_{q^2,-1}(b(X))) \\ &= \gamma a(X) \tau_{q^2,-1}(b(X)) - \gamma \tau_{q,1}(a(X) \tau_{q^2,-1}(b(X))). \end{aligned}$$

So for $0 \leq k \leq q$,

$$[\rho_{i,0}(X)^k, \rho_{i,0}(X)^k]_{\Delta} = \gamma \rho_{i,0}(X)^k \rho_{i,0}(X)^{kq^2} - \gamma \rho_{i,0}(X)^{kq} \rho_{i,0}(X)^{kq^3}.$$

As $\rho_{i,0}(X)^{q^2} = \rho_{i,0}(X) \in \mathcal{I}_{i,0} \cong \mathbb{F}_{q^2}$, we have

$$[\rho_{i,0}(X)^k, \rho_{i,0}(X)^k]_{\Delta} = \gamma \rho_{i,0}(X)^k \rho_{i,0}(X)^k - \gamma \rho_{i,0}(X)^{kq} \rho_{i,0}(X)^{kq} = 0$$

if and only if $\rho_{i,0}(X)^{2k(q-1)} = e_{i,0}(X)$ if and only if $2k(q-1) \equiv 0 \pmod{q^2-1}$ if and only if $2k \equiv 0 \pmod{q+1}$. When q is even, we have $k \equiv 0 \pmod{q+1}$. Thus, we have $k = 0$. When q is odd, we have $k \equiv 0 \pmod{\frac{q+1}{2}}$. Since $0 \leq k \leq q$, we have $k = \frac{q+1}{2}$.

(ii) Now consider the case when $\mu(i) = i$ but $i \notin \{0, i^{\#}\}$. By Lemma 6 (i) and Lemma 2, we have d_i is even implying $s_i = \gcd(2, d_i) = 2$ and $tD_i = 2D_i = d_i$ as $D_i = d_i/s_i$. Hence, by Lemma 3, $\mathcal{J}_i = \mathcal{I}_{i,0} \oplus \mathcal{I}_{i,1}$ and $\tau_{q,1}(\mathcal{I}_{i,0}) = \mathcal{I}_{i,1}$. Using Lemma 2 (iv) and (v), we have $\mathcal{I}_{i,0} \cong \mathcal{I}_{i,1} \cong \mathcal{K}_i \cong \mathbb{F}_{q^{d_i}}$. As in part (i), we need the bases of the different 1-dimensional \mathcal{K}_i -subspaces of \mathcal{J}_i . Since $\tau_{q^2,1}$ is the identity on $\mathcal{R}_n^{(q^2)}$, by Lemma 4, we have $\mathcal{K}_i = \{c(X) + \tau_{q,1}(c(X)) | c(X) \in \mathcal{I}_{i,0}\} \cong \mathbb{F}_{q^{d_i}}$. Therefore, the bases of the $q^{d_i} + 1$ 1-dimensional \mathcal{K}_i -subspaces of \mathcal{J}_i are $\{e_{i,0}(X)\}$, $\{e_{i,1}(X)\}$ and $\{e_{i,0}(X) + \rho_{i,1}(X)^k\}$, where $0 \leq k \leq q^{d_i} - 2$.

According to part (i), if \mathcal{C}_i is 1-dimensional over \mathcal{K}_i , then $\mathcal{C}_i = \mathcal{C}_i^{(\Delta)}$. Assume that $\tau_{1,-1}(\mathcal{I}_{i,j}) = \mathcal{I}_{i,j}$ for $0 \leq j \leq 1$. By Lemma 7 (iii), we have $\tau_{1,-1}(c(X)) = c(X)^{q^{tD_i/2}} = c(X)^{q^{d_i/2}}$ for $c(X)$ in either $\mathcal{I}_{i,0}$ or $\mathcal{I}_{i,1}$. Recall that

$$[a(X), b(X)]_{\Delta} = \gamma a(X) \tau_{q^2,-1}(b(X)) - \gamma \tau_{q,1}(a(X) \tau_{q^2,-1}(b(X))).$$

Firstly, since

$$\begin{aligned} [e_{i,0}(X), e_{i,0}(X)]_{\Delta} &= \gamma e_{i,0}(X) \tau_{q^2,-1}(e_{i,0}(X)) - \gamma \tau_{q,1}(e_{i,0}(X) \tau_{q^2,-1}(e_{i,0}(X))) \\ &= \gamma e_{i,0}(X) e_{i,0}(X)^{q^{d_i/2}} - \gamma e_{i,1}(X) e_{i,1}(X)^{q^{d_i/2}} \\ &= \gamma e_{i,0}(X) - \gamma e_{i,1}(X) \neq 0. \end{aligned}$$

Similarly, we have $[e_{i,1}(X), e_{i,1}(X)]_{\Delta} \neq 0$. So neither $\{e_{i,0}(X)\}$ nor $\{e_{i,1}(X)\}$ is a basis of \mathcal{C}_i . Secondly, we have

$$\begin{aligned} [e_{i,0}(X) + \rho_{i,1}(X)^k, e_{i,0}(X) + \rho_{i,1}(X)^k]_{\Delta} &= \gamma (e_{i,0}(X) + \rho_{i,1}(X)^k) (e_{i,0}(X)^{q^{d_i/2}} + \rho_{i,1}(X)^{kq^{d_i/2}}) \\ &\quad - \gamma (e_{i,1}(X) + \rho_{i,0}(X)^k) (e_{i,1}(X)^{q^{d_i/2}} + \rho_{i,0}(X)^{kq^{d_i/2}}) \\ &= \gamma (e_{i,0}(X) + \rho_{i,1}(X)^{k(q^{d_i/2}+1)}) - \gamma (e_{i,1}(X) + \rho_{i,0}(X)^{k(q^{d_i/2}+1)}) \\ &= \gamma (e_{i,0}(X) - \rho_{i,0}(X)^{k(q^{d_i/2}+1)}) - \gamma (e_{i,1}(X) - \rho_{i,1}(X)^{k(q^{d_i/2}+1)}). \end{aligned}$$

Then, we have $[e_{i,0}(X) + \rho_{i,1}(X)^k, e_{i,0}(X) + \rho_{i,1}(X)^k]_{\Delta} = 0$ if and only if $\rho_{i,0}(X)^{k(q^{d_i/2}+1)} = e_{i,0}(X)$ and $\rho_{i,1}(X)^{k(q^{d_i/2}+1)} = e_{i,1}(X)$ if and only if $k(q^{d_i/2} + 1) \equiv 0 \pmod{q^{d_i} - 1}$ if and only if $k \equiv 0 \pmod{q^{d_i/2} - 1}$ if and only if $k = (q^{d_i/2} - 1)m$ for $0 \leq m \leq q^{d_i/2}$, since d_i is even and $0 \leq k \leq q^{d_i} - 2$.

(iii) By part (ii), assume that $\tau_{1,-1}(\mathcal{I}_{i,0}) = \mathcal{I}_{i,1}$. As $\tau_{q,1}(\mathcal{I}_{i,0}) = \mathcal{I}_{i,1}$, $\tau_{q,-1}(\mathcal{I}_{i,j}) = \mathcal{I}_{i,j}$. Recall that, $\tau_{q^2,1}$ is the identity on $\mathcal{R}_n^{(q^2)}$,

$$\begin{aligned} [a(X), b(X)]_\Delta &= \gamma a(X) \tau_{q^2,-1}(b(X)) - \gamma \tau_{q,1}(a(X) \tau_{q^2,-1}(b(X))) \\ &= \gamma a(X) \tau_{1,-1}(b(X)) - \gamma \tau_{q,1}(a(X) \tau_{1,-1}(b(X))). \end{aligned}$$

It is not difficult to verify that, by Lemma 2 (v), $[e_{i,j}(X), e_{i,j}(X)]_\Delta = 0$ for $0 \leq j \leq 1$. So \mathcal{C}_i is 1-dimensional over \mathcal{K}_i with bases $\{e_{i,0}(X)\}$ and $\{e_{i,1}(X)\}$. Since

$$\begin{aligned} [e_{i,0}(X) + \rho_{i,1}(X)^k, e_{i,0}(X) + \rho_{i,1}(X)^k]_\Delta &= \gamma(e_{i,0}(X) + \rho_{i,1}(X)^k)(e_{i,1}(X)^{q^{d_i/2}} + \rho_{i,0}(X)^{kq^{d_i/2}}) \\ &\quad - \gamma \tau_{q,1}((e_{i,0}(X) + \rho_{i,1}(X)^k)(e_{i,1}(X)^{q^{d_i/2}} + \rho_{i,0}(X)^{kq^{d_i/2}})) \\ &= \gamma(\rho_{i,0}(X)^{kq^{d_i/2}} + \rho_{i,1}(X)^k) - \gamma(\rho_{i,1}(X)^{kq^{d_i/2}} + \rho_{i,0}(X)^k) \\ &= \gamma(\rho_{i,0}(X)^{kq^{d_i/2}} - \rho_{i,0}(X)^k) - \gamma(\rho_{i,1}(X)^{kq^{d_i/2}} - \rho_{i,1}(X)^k), \end{aligned}$$

$[e_{i,0}(X) + \rho_{i,1}(X)^k, e_{i,0}(X) + \rho_{i,1}(X)^k]_\Delta = 0$ if and only if $\rho_{i,0}(X)^{k(q^{d_i/2}-1)} = e_{i,0}(X)$ and $\rho_{i,1}(X)^{k(q^{d_i/2}-1)} = e_{i,1}(X)$ if and only if $k(q^{d_i/2} - 1) \equiv 0 \pmod{q^{d_i} - 1}$ if and only if $k \equiv 0 \pmod{q^{d_i/2} + 1}$ if and only if $k = (q^{d_i/2} + 1)m$, where $0 \leq m \leq q^{d_i/2} - 2$ as $0 \leq k \leq q^{d_i} - 2$.

(iv) Now, suppose that $\mu(i) \neq i$ and d_i is odd. By Lemmas 2 and 4, we have $s_i = \gcd(2, d_i) = 1$, $\mathcal{J}_i = \mathcal{I}_{i,0} \cong \mathbb{F}_{q^{2D_i}} = \mathbb{F}_{q^{2d_i}}$ and $\mathcal{K}_i = \{c(X) \in \mathcal{I}_{i,0} | \tau_{q,1}(c(X)) = c(X)\} \cong \mathbb{F}_{q^{d_i}}$. As $\tau_{1,-1}$ is an isomorphism of \mathcal{J}_i onto $\mathcal{J}_{\mu(i)}$, we have $d_{\mu(i)} = d_i$, $s_{\mu(i)} = \gcd(2, d_{\mu(i)}) = 1$, $\mathcal{J}_{\mu(i)} = \mathcal{I}_{\mu(i),0} \cong \mathbb{F}_{q^{2d_i}}$ and $\mathcal{K}_{\mu(i)} = \{c(X) \in \mathcal{I}_{\mu(i),0} | \tau_{q,1}(c(X)) = c(X)\} \cong \mathbb{F}_{q^{d_i}}$. $\tau_{q^2,1}$ is the identity on $\mathcal{R}_n^{(q^2)}$ and hence $\tau_{q,1} : \mathcal{J}_i \rightarrow \mathcal{J}_i$ is the identity on \mathcal{K}_i but not \mathcal{J}_i by Lemma 4. Thus, $\tau_{q,1}$ is an automorphism of \mathcal{J}_i of order 2 implying $\tau_{q,1}(c(X)) = c(X)^{q^{2d_i/2}} = c(X)^{q^{d_i}}$ for all $c(X) \in \mathcal{J}_i$. Similarly, $\tau_{q,1}(c(X)) = c(X)^{q^{d_i}}$ for all $c(X) \in \mathcal{J}_{\mu(i)}$.

By Theorem 1, we have

$$\mathcal{C}_{\mu(i)}^{(\Delta)} = \{a(X) \in \mathcal{J}_{\mu(i)} | [c(X), a(X)]_\Delta = 0 \text{ for all } c(X) \in \mathcal{C}_i\}$$

and $\dim_{\mathcal{K}_i} \mathcal{C}_i + \dim_{\mathcal{K}_{\mu(i)}} \mathcal{C}_{\mu(i)}^{(\Delta)} = t = 2$. As $\mathcal{C}_{\mu(i)} \subseteq \mathcal{C}_{\mu(i)}^{(\Delta)}$ is required for cyclic Δ -self-orthogonality, for each possible \mathcal{C}_i , we must find $\mathcal{C}_{\mu(i)}^{(\Delta)}$. It is easy to show that, as the dimensions add to 2, $\mathcal{C}_i = \{0\}$ implies $\mathcal{C}_{\mu(i)}^{(\Delta)} = \mathcal{J}_{\mu(i)}$ and $\mathcal{C}_i = \mathcal{J}_i$ implies $\mathcal{C}_{\mu(i)}^{(\Delta)} = \{0\}$. If \mathcal{C}_i is 1-dimensional over \mathcal{K}_i , then $\mathcal{C}_{\mu(i)}^{(\Delta)}$ is 1-dimensional over $\mathcal{K}_{\mu(i)}$. Furthermore, if \mathcal{C}_i is 1-dimensional over \mathcal{K}_i , then \mathcal{C}_i has basis $\{\rho_{i,0}(X)^k\}$ for some k . As \mathcal{K}_i has primitive element $\rho_{i,0}(X)^{q^{d_i}+1}$, \mathcal{C}_i has $q^{d_i} + 1$ different bases $\{\rho_{i,0}(X)^k\}$ for $0 \leq k \leq q^{d_i}$. Now we find k' , where $0 \leq k' \leq q^{d_i}$, such that $[\rho_{i,0}(X)^k, \rho_{\mu(i),0}(X)^{k'}]_\Delta = 0$. Using $\tau_{1,-1}(\rho_{i,0}(X)) =$

$\rho_{\mu(i),0}(X)$ and $\tau_{q,1}(c(X)) = c(X)^{q^{d_i}}$ for $c(X) \in \mathcal{J}_i$ or $\mathcal{J}_{\mu(i)}$, we have

$$\begin{aligned} & [\rho_{i,0}(X)^k, \rho_{\mu(i),0}(X)^{k'}]_{\Delta} \\ &= \gamma \rho_{i,0}(X)^k \tau_{1,-1}(\rho_{\mu(i),0}(X)^{k'}) - \gamma \tau_{q,1}(\rho_{i,0}(X)^k \tau_{1,-1}(\rho_{\mu(i),0}(X)^{k'})) \\ &= \gamma \rho_{i,0}(X)^{k+k'} - \gamma \rho_{i,0}(X)^{(k+k')q^{d_i}} \\ &= 0 \end{aligned}$$

if and only if $\rho_{i,0}(X)^{(k+k')(q^{d_i}-1)} = e_{i,0}(X)$ if and only if $(k+k')(q^{d_i}-1) \equiv 0 \pmod{q^{2d_i}-1}$ if and only if $k+k' \equiv 0 \pmod{q^{d_i}+1}$ if and only if $k' = k = 0$ or $k' = q^{d_i} + 1 - k$ for $1 \leq k \leq q^{d_i}$.

(v) Finally, assume that $\mu(i) \neq i$ and d_i is even. From the proof of parts (ii)-(iv), we have $s_i = 2$, $\mathcal{J}_i = \mathcal{I}_{i,0} \oplus \mathcal{I}_{i,1}$, $\mathcal{I}_{i,0} \cong \mathcal{I}_{i,1} \cong \mathbb{F}_{q^{d_i}}$, $\mathcal{K}_i = \{c(X) + \tau_{q,1}(c(X)) | c(X) \in \mathcal{I}_{i,0}\} \cong \mathbb{F}_{q^{d_i}}$ and the $q^{d_i} + 1$ 1-dimensional \mathcal{K}_i -subspaces of \mathcal{J}_i have bases $\{e_{i,0}(X)\}$, $\{e_{i,1}(X)\}$ and $\{e_{i,0}(X) + \rho_{i,1}(X)^k\}$, where $0 \leq k \leq q^{d_i} - 2$. In this case, $\tau_{1,-1}(\rho_{i,j}(X)) = \rho_{\mu(i),j}(X)$ for $0 \leq j \leq 1$, $\tau_{q,1}(\rho_{i,0}(X)) = \rho_{i,1}(X)$ and $\tau_{q,1}(\rho_{\mu(i),0}(X)) = \rho_{\mu(i),1}(X)$.

If $\mathcal{C}_i = \{0\}$ or $\mathcal{C}_i = \mathcal{J}_i$, (a) and (b) follow as in the proof of part (iv). Now we consider the basis of $\mathcal{C}_{\mu(i)}^{(\Delta)}$ when \mathcal{C}_i is 1-dimensional over \mathcal{K}_i . Then $\mathcal{C}_{\mu(i)}^{(\Delta)}$ will be 1-dimensional over $\mathcal{K}_{\mu(i)}$. As

$$[a(X), b(X)]_{\Delta} = \gamma a(X) \tau_{1,-1}(b(X)) - \gamma \tau_{q,1}(a(X) \tau_{1,-1}(b(X))),$$

we have

$$\begin{aligned} & [e_{i,0}(X), e_{\mu(i),0}(X)]_{\Delta} \\ &= \gamma e_{i,0}(X) \tau_{1,-1}(e_{\mu(i),0}(X)) - \gamma \tau_{q,1}(e_{i,0}(X) \tau_{1,-1}(e_{\mu(i),0}(X))) \\ &= \gamma e_{i,0}(X) - \gamma e_{i,1}(X) \neq 0. \end{aligned}$$

Similarly, $[e_{i,1}(X), e_{\mu(i),1}(X)]_{\Delta} \neq 0$. So neither $\{e_{i,0}(X)\}$ nor $\{e_{i,1}(X)\}$ is a basis of \mathcal{C}_i . Now we find k' , where $0 \leq k' \leq q^{d_i} - 2$, such that

$$[e_{i,0}(X) + \rho_{i,1}(X)^k, e_{\mu(i),0}(X) + \rho_{\mu(i),1}(X)^{k'}]_{\Delta} = 0.$$

By $\mathcal{I}_{i,0} \mathcal{I}_{i,1} = \{0\}$, we have

$$\begin{aligned} & [e_{i,0}(X) + \rho_{i,1}(X)^k, e_{\mu(i),0}(X) + \rho_{\mu(i),1}(X)^{k'}]_{\Delta} \\ &= \gamma(e_{i,0}(X) + \rho_{i,1}(X)^k) \tau_{1,-1}(e_{\mu(i),0}(X) + \rho_{\mu(i),1}(X)^{k'}) \\ &\quad - \gamma \tau_{q,1}((e_{i,0}(X) + \rho_{i,1}(X)^k) \tau_{1,-1}(e_{\mu(i),0}(X) + \rho_{\mu(i),1}(X)^{k'})) \\ &= \gamma(e_{i,0}(X) + \rho_{i,1}(X)^k)(e_{i,0}(X) + \rho_{i,1}(X)^{k'}) \\ &\quad - \gamma(e_{i,1}(X) + \rho_{i,0}(X)^k)(e_{i,1}(X) + \rho_{i,0}(X)^{k'}) \\ &= \gamma(e_{i,0}(X) + \rho_{i,1}(X)^{k+k'}) - \gamma(e_{i,1}(X) + \rho_{i,0}(X)^{k+k'}) \\ &= \gamma(e_{i,0}(X) - \rho_{i,0}(X)^{k+k'}) - \gamma(e_{i,1}(X) - \rho_{i,1}(X)^{k+k'}) \\ &= 0 \end{aligned}$$

if and only if $\rho_{i,0}(X)^{k+k'} = e_{i,0}(X)$ and $\rho_{i,1}(X)^{k+k'} = e_{i,1}(X)$ if and only if $k+k' \equiv 0 \pmod{q^{d_i}-1}$ if and only if $k' = k = 0$ or $k' = q^{d_i} - 1 - k$ for $1 \leq k \leq q^{d_i} - 2$.

When $t = 2$, we can count the number of cyclic Δ -self-orthogonal \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes. Let \mathfrak{J} be the fixed points of μ excluding 0 and $i^\#$ (when n is even). Let \mathfrak{M} consist of one element from each of the transpositions in μ . We give the number of cyclic Δ -self-orthogonal \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes in the following Theorem.

Theorem 3 *When $t = 2$, the number of cyclic Δ -self-orthogonal \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes of length n is as follows, where $\gcd(n, q) = 1$.*

$$a' \prod_{i \in \mathfrak{J}} (q^{d_i/2} + 2) \prod_{j \in \mathfrak{M}} (3q^{d_j} + b'),$$

where $a' = 2$ with n odd, $a' = 4$ with n even, $b' = 6$ if d_i is odd and $b' = 2$ if d_i is even.

Proof The trick of the proof is to count the number of subcodes \mathcal{C}_i when $\mu(i) = i$ and subcodes pairs $(\mathcal{C}_i, \mathcal{C}_{\mu(i)})$ when $\mu(i) \neq i$ in Theorem 2. Let M_i be the number of distinct \mathcal{K}_i -subspaces \mathcal{C}_i of \mathcal{J}_i satisfying $\mathcal{C}_i \subseteq \mathcal{C}_i^{(\Delta)}$, where $i \in \mathfrak{J}$. Let M_j be the number of distinct pairs $(\mathcal{C}_j, \mathcal{C}_{\mu(j)})$, where $j \in \mathfrak{M}$, \mathcal{C}_j is a \mathcal{K}_j -subspace of \mathcal{J}_j and $\mathcal{C}_{\mu(j)}$ is a $\mathcal{K}_{\mu(j)}$ -subspace of $\mathcal{J}_{\mu(j)}$, satisfying $\mathcal{C}_j \subseteq \mathcal{C}_j^{(\Delta)}$ and $\mathcal{C}_{\mu(j)} \subseteq \mathcal{C}_{\mu(j)}^{(\Delta)}$. Then, by Theorem 2, the number of distinct cyclic Δ -self-orthogonal \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes of length n is

$$\begin{cases} 4 \prod_{i \in \mathfrak{J}} M_i \prod_{j \in \mathfrak{M}} M_j, & \text{if } n \text{ is even,} \\ 2 \prod_{i \in \mathfrak{J}} M_i \prod_{j \in \mathfrak{M}} M_j, & \text{if } n \text{ is odd.} \end{cases}$$

It is straightforward to show that $M_i = q^{d_i/2} + 2$ for $i \in \mathfrak{J}$, $M_j = 3q^{d_j} + 6$ for d_i is odd and $j \in \mathfrak{M}$ and $M_j = 3q^{d_j} + 2$ for d_i is even and $j \in \mathfrak{M}$. The proof is completed.

4.2 Cyclic Δ -self-dual \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes

Using Theorem 2, we determine the bases of all cyclic Δ -self-dual \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes of length n in the following Theorem. By Theorem 1, $\dim_{\mathcal{K}_i} \mathcal{C}_i + \dim_{\mathcal{K}_{\mu(i)}} \mathcal{C}_{\mu(i)} = t = 2$ for $0 \leq i \leq s-1$. In particular, if $\mu(i) = i$, then $\dim_{\mathcal{K}_i} \mathcal{C}_i = 1$ and if $\mu(i) \neq i$, by Theorem 2, we have $\mathcal{C}_{\mu(i)} = \mathcal{C}_{\mu(i)}^{(\Delta)}$.

Theorem 4 *Let \mathcal{C} be a cyclic \mathbb{F}_q -linear \mathbb{F}_{q^2} -code of length n . If $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \cdots \oplus \mathcal{C}_{s-1}$, where $\mathcal{C}_i = \mathcal{C} \cap \mathcal{J}_i$ for all $0 \leq i \leq s-1$, then \mathcal{C} is cyclic Δ -self-dual when the following hold.*

- (i) *When $i = 0$ or, if n is even, $i = i^\#$, then \mathcal{C}_i is a 1-dimensional \mathcal{K}_i -subspace of \mathcal{J}_i with basis $\{\rho_{i,0}(X)^k\}$, where $k = 0$ when q is even and $k = \frac{q+1}{2}$ when q is odd.*
- (ii) *When $i \neq 0$, $i \neq i^\#$, $\mu(i) = i$ and $\tau_{1,-1}(\mathcal{I}_{i,0}) = \mathcal{I}_{i,0}$, then \mathcal{C}_i is 1-dimensional over \mathcal{K}_i with basis $\{e_{i,0}(X) + \rho_{i,1}(X)^k\}$, where $k = (q^{d_i/2} - 1)m$ for $0 \leq m \leq q^{d_i/2}$.*

- (iii) When $i \neq 0$, $i \neq i^\#$, $\mu(i) = i$ and $\tau_{1,-1}(\mathcal{I}_{i,0}) = \mathcal{I}_{i,1}$, then \mathcal{C}_i is 1-dimensional over \mathcal{K}_i with bases $\{e_{i,0}(X)\}$, $\{e_{i,1}(X)\}$ and $\{e_{i,0}(X) + \rho_{i,1}(X)^k\}$, where $k = (q^{d_i/2} + 1)m$ for $0 \leq m \leq q^{d_i/2} - 2$.
- (iv) When $\mu(i) \neq i$ and d_i is odd,
- (a) if $\mathcal{C}_i = \{0\}$, then $\mathcal{C}_{\mu(i)} = \mathcal{J}_{\mu(i)}$;
 - (b) if $\mathcal{C}_i = \mathcal{J}_i$, then $\mathcal{C}_{\mu(i)} = \{0\}$;
 - (c) if \mathcal{C}_i is 1-dimensional over \mathcal{K}_i with basis $\{\rho_{i,0}(X)^k\}$, where $0 \leq k \leq q^{d_i}$, then $\mathcal{C}_{\mu(i)}$ is 1-dimensional over $\mathcal{K}_{\mu(i)}$ with basis $\{\rho_{\mu(i),0}(X)^{k'}\}$, where $k' = k = 0$ or $k' = q^{d_i} + 1 - k$ for $1 \leq k \leq q^{d_i}$.
- (v) When $\mu(i) \neq i$ and d_i is even,
- (a) if $\mathcal{C}_i = \{0\}$, then $\mathcal{C}_{\mu(i)} = \mathcal{J}_{\mu(i)}$;
 - (b) if $\mathcal{C}_i = \mathcal{J}_i$, then $\mathcal{C}_{\mu(i)} = \{0\}$;
 - (c) if \mathcal{C}_i is 1-dimensional over \mathcal{K}_i with basis $\{a(X)\}$, then $\mathcal{C}_{\mu(i)}$ is 1-dimensional over $\mathcal{K}_{\mu(i)}$ with basis $\{a'(X)\}$, where $a'(X) = e_{\mu(i),0}(X) + \rho_{\mu(i),1}(X)^{k'}$ when $a(X) = e_{i,0}(X) + \rho_{i,1}(X)^k$ with $k' = k = 0$ or $k' = q^{d_i} - 1 - k$ for $1 \leq k \leq q^{d_i} - 2$.

Proof The result follows immediately by Theorem 2.

When $t = 2$, similarly, we give the number of cyclic Δ -self-dual \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes in the following Theorem.

Theorem 5 When $t = 2$, the number of cyclic Δ -self-dual \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes of length n is as follows, where $\gcd(n, q) = 1$.

$$\prod_{i \in \mathfrak{J}} (q^{d_i/2} + 1) \prod_{j \in \mathfrak{M}} (q^{d_j} + b'),$$

where $b' = 3$ if d_i is odd and $b' = 1$ if d_i is even.

Proof The proof of this result is quite similar to Theorem 3, and so is omitted.

5 An example

In this case, we have $q = p = 3$, $t = 2$ and $n = 7$. Then $t = 2^a m$ and $Q = 2^{a-1} = 1$, where $a = 1$, $m = 1$. Let $\mathbb{F}_{3^2} = \mathbb{F}_3[X]/\langle X^2 + 2X + 2 \rangle$, where $X^2 + 2X + 2$ is a primitive polynomial over $\mathbb{F}_3[X]$. Let ω be a root of $X^2 + 2X + 2$, it is trivial to see that ω is a primitive element of \mathbb{F}_{3^2} with $\text{ord}(\omega) = 3^2 - 1 = 8$. Thus $\mathcal{R}_7^{(3)} = \mathbb{F}_3[X]/\langle X^7 - 1 \rangle$ and $\mathcal{R}_7^{(3^2)} = \mathbb{F}_{3^2}[X]/\langle X^7 - 1 \rangle$. All distinct cyclic Δ -self-orthogonal and cyclic Δ -self-dual \mathbb{F}_3 -linear \mathbb{F}_{3^2} -codes of length 7 and their enumeration can be given in the following Steps.

Step 1 Factorize $X^7 - 1 = \prod_{i=0}^1 m_i(X)$, where $m_0(X) = X - 1 = X + 2$ and $m_1(X) = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$ are monic irreducible polynomials over $\mathbb{F}_3[X]$. Let $\mathbb{F}_{3^6} = \mathbb{F}_3[X]/\langle X^6 + 2X^4 + X^2 + 2X + 2 \rangle$, where $X^6 + 2X^4 + X^2 + 2X + 2$ is a primitive polynomial over $\mathbb{F}_3[X]$, and η be a root of $X^6 + 2X^4 + X^2 + 2X + 2$, then we have $\text{ord}(\eta) = 3^6 - 1 = 728$. Thus $\eta' = \eta^{104}$ is a primitive 7th root of unity over \mathbb{F}_{3^6} .

According to Lemma 2 (i), as $C_0^{(3)} = \{0\}$ and $C_1^{(3)} = \{1, 2, 3, 4, 5, 6\}$, it is not difficult to find that $l_0 = 0$, $l_1 = 1$, $d_0 = \deg m_0(X) = |C_{l_0}^{(3)}| = 1$ and $d_1 = \deg m_1(X) = |C_{l_1}^{(3)}| = 6$. Furthermore, we have $m_0(X) \leftrightarrow C_{l_0}^{(3)} = C_0^{(3)}$ and $m_1(X) \leftrightarrow C_{l_1}^{(3)} = C_1^{(3)}$. That is,

$$m_0(X) = X - 1 = X - (\eta^{104})^0$$

and

$$\begin{aligned} m_1(X) &= X^6 + X^5 + X^4 + X^3 + X^2 + X + 1 \\ &= (X - (\eta^{104})^1)(X - (\eta^{104})^2)(X - (\eta^{104})^3)(X - (\eta^{104})^4) \\ &\quad \cdot (X - (\eta^{104})^5)(X - (\eta^{104})^6) \\ &= (X - \eta^{104})(X - \eta^{208})(X - \eta^{312})(X - \eta^{416})(X - \eta^{520})(X - \eta^{624}). \end{aligned}$$

Step 2 By computer systems Maple and Magma (<http://magma.maths.usyd.edu.au/calc/>), it is easy to verify that $m_0(X) = X - 1 = M_{0,0}(X)$ and $m_1(X) = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1 = M_{1,0}(X)M_{1,1}(X)$, where $M_{1,0}(X) = X^3 + \omega X^2 + \omega^7 X + 2$, $M_{1,1}(X) = X^3 + \omega^3 X^2 + \omega^5 X + 2$ and $M_{0,0}(X)$, $M_{1,0}(X)$, $M_{1,1}(X)$ are monic irreducible polynomials over $\mathbb{F}_{3^2}[X]$. So, by Lemma 2 (ii) and (iii), we have $s_0 = 1$, $s_1 = 2$ and $X^7 - 1 = \prod_{i=0}^1 \prod_{j=0}^{s_i-1} M_{i,j}(X)$.

Since $M_{i,j}(X) \leftrightarrow C_{l_i 3^j}^{(3^2)}$, $M_{0,0}(X) \leftrightarrow C_{l_0 3^0}^{(3^2)} = C_0^{(9)} = \{0\}$, $M_{1,0}(X) \leftrightarrow C_{l_1 3^0}^{(3^2)} = C_1^{(9)} = \{1, 2, 4\}$ and $M_{1,1}(X) \leftrightarrow C_{l_1 3^1}^{(3^2)} = C_3^{(9)} = \{3, 5, 6\}$. That is,

$$M_{0,0}(X) = X - 1 = X - (\eta^{104})^0,$$

$$\begin{aligned} M_{1,0}(X) &= X^3 + \omega X^2 + \omega^7 X + 2 = (X - (\eta^{104})^1)(X - (\eta^{104})^2)(X - (\eta^{104})^4) \\ &= (X - \eta^{104})(X - \eta^{208})(X - \eta^{416}) \end{aligned}$$

and

$$\begin{aligned} M_{1,1}(X) &= X^3 + \omega^3 X^2 + \omega^5 X + 2 = (X - (\eta^{104})^3)(X - (\eta^{104})^5)(X - (\eta^{104})^6) \\ &= (X - \eta^{312})(X - \eta^{520})(X - \eta^{624}). \end{aligned}$$

Step 3 By Lemma 2 (iv) and (v), we have $\mathcal{R}_7^{(3)} = \mathcal{K}_0 \oplus \mathcal{K}_1$, where \mathcal{K}_0 is the ideal of $\mathcal{R}_7^{(3)}$ generated by $\hat{m}_0(X) = (X^7 - 1)/m_0(X) = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$ denoted by $\mathcal{K}_0 = \langle X^6 + X^5 + X^4 + X^3 + X^2 + X + 1 \rangle \cong \mathbb{F}_{3^{d_0}} = \mathbb{F}_3$. Similarly, we have $\mathcal{K}_1 = \langle X + 2 \rangle \cong \mathbb{F}_{3^{d_1}} = \mathbb{F}_{3^6}$.

As $D_i = d_i/s_i$ for $0 \leq i \leq 1$, $D_0 = 1$ and $D_1 = 3$. Then, similarly, we have $\mathcal{R}_7^{(3^2)} = \mathcal{I}_{0,0} \oplus \mathcal{I}_{1,0} \oplus \mathcal{I}_{1,1}$, where $\mathcal{I}_{0,0} = \langle (X^7 - 1)/M_{0,0}(X) \rangle = \langle X^6 + X^5 + X^4 + X^3 + X^2 + X + 1 \rangle \cong \mathbb{F}_{3^{2D_0}} = \mathbb{F}_{3^2}$, $\mathcal{I}_{1,0} = \langle X^4 + \omega^5 X^3 + 2X^2 + \omega^7 X + 1 \rangle \cong \mathbb{F}_{3^{2D_1}} = \mathbb{F}_{3^6}$ and $\mathcal{I}_{1,1} = \langle X^4 + \omega^7 X^3 + 2X^2 + \omega^5 X + 1 \rangle \cong \mathbb{F}_{3^{2D_1}} = \mathbb{F}_{3^6}$. In addition, we have $\mathcal{J}_0 = \mathcal{I}_{0,0}$ and $\mathcal{J}_1 = \mathcal{I}_{1,0} \oplus \mathcal{I}_{1,1}$.

Step 4 By Wan [13] and computer systems Maple and Magma, we can obtain non-zero idempotents $e_{i,j}(X)$ of $\mathcal{I}_{i,j}$, where $0 \leq i \leq 1$ and $0 \leq j \leq s_i - 1$. That is,

$$e_{0,0}(X) = X^6 + X^5 + X^4 + X^3 + X^2 + X + 1,$$

$$e_{1,0}(X) = \omega^5 X^6 + \omega^5 X^5 + \omega^7 X^4 + \omega^5 X^3 + \omega^7 X^2 + \omega^7 X$$

and

$$e_{1,1}(X) = \omega^7 X^6 + \omega^7 X^5 + \omega^5 X^4 + \omega^7 X^3 + \omega^5 X^2 + \omega^5 X.$$

It is easy to show that $e_{i,j}(X)$ is the identity of $\mathcal{I}_{i,j}$ for $0 \leq i \leq 1$ and $0 \leq j \leq s_i - 1$.

It is easy to verify that η and η^{243} are primitive elements of finite field \mathbb{F}_{3^6} . As $\mathcal{I}_{0,0} \cong \mathbb{F}_{3^2}$, $\mathcal{I}_{1,0} \cong \mathbb{F}_{3^6}$ and $\mathcal{I}_{1,1} \cong \mathbb{F}_{3^6}$, we choose

$$\rho_{0,0}(X) = \omega \cdot e_{0,0}(X) = \omega X^6 + \omega X^5 + \omega X^4 + \omega X^3 + \omega X^2 + \omega X + \omega,$$

$$\begin{aligned} \rho_{1,0}(X) &= \eta^{243} \cdot e_{1,0}(X) \\ &= \eta^{243} \omega^5 X^6 + \eta^{243} \omega^5 X^5 + \eta^{243} \omega^7 X^4 + \eta^{243} \omega^5 X^3 + \eta^{243} \omega^7 X^2 + \eta^{243} \omega^7 X \end{aligned}$$

and

$$\rho_{1,1}(X) = \eta \cdot e_{1,1}(X) = \eta \omega^7 X^6 + \eta \omega^7 X^5 + \eta \omega^5 X^4 + \eta \omega^7 X^3 + \eta \omega^5 X^2 + \eta \omega^5 X$$

are primitive elements of $\mathcal{I}_{i,j}$ satisfying $\tau_{3^j,1}(\rho_{i,0}(X)) = \rho_{i,j}(X)$ for $0 \leq i \leq 1$ and $0 \leq j \leq s_i - 1$.

Step 5 It is easy to show that $C_{-l_1}^{(3)} = C_{-1}^{(3)} = \{1, 2, 3, 4, 5, 6\} = C_1^{(3)} = C_{l_1}^{(3)} = C_{l_{\mu(1)}}^{(3)}$, then $\mu(1) = 1$. By Maple and Magma, we verify that $\tau_{1,-1}(\mathcal{I}_{1,0}) = \mathcal{I}_{1,1}$.

Let \mathcal{C} be a cyclic \mathbb{F}_3 -linear \mathbb{F}_{3^2} -code of length 7 and $\mathcal{C}^{\perp \Delta}$ the dual code of \mathcal{C} . We write $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ and $\mathcal{C}^{\perp \Delta} = \mathcal{C}_0^{(\Delta)} \oplus \mathcal{C}_1^{(\Delta)}$, where $\mathcal{C}_i = \mathcal{C} \cap \mathcal{J}_i$ and $\mathcal{C}_i^{(\Delta)} = \mathcal{C}^{\perp \Delta} \cap \mathcal{J}_i$ for all $0 \leq i \leq 1$. According to Theorem 2, \mathcal{C} is cyclic Δ -self-orthogonal if and only if for each i ($0 \leq i \leq 1$), the following assertions hold.

- (i) When $i = 0$, then
 - (a) $\mathcal{C}_0 = \{0\}$, or
 - (b) \mathcal{C}_0 is 1-dimensional \mathcal{K}_0 -subspace of \mathcal{J}_0 with basis $\{\rho_{0,0}(X)^2\} = \{\omega^2 X^6 + \omega^2 X^5 + \omega^2 X^4 + \omega^2 X^3 + \omega^2 X^2 + \omega^2 X + \omega^2\}$.
- (ii) When $i \neq 0$, we have $\mu(1) = 1$ and $\tau_{1,-1}(\mathcal{I}_{1,0}) = \mathcal{I}_{1,1}$. Then
 - (a) $\mathcal{C}_1 = \{0\}$, or
 - (b) \mathcal{C}_1 is 1-dimensional over \mathcal{K}_1 with bases $\{e_{1,0}(X)\} = \{\omega^5 X^6 + \omega^5 X^5 + \omega^7 X^4 + \omega^5 X^3 + \omega^7 X^2 + \omega^7 X\}$, $\{e_{1,1}(X)\} = \{\omega^7 X^6 + \omega^7 X^5 + \omega^5 X^4 + \omega^7 X^3 + \omega^5 X^2 + \omega^5 X\}$ and $\{e_{1,0}(X) + \rho_{1,1}(X)^k\}$, where $k = 28m$ for $0 \leq m \leq 25$.

According to Theorem 3, the number of cyclic Δ -self-orthogonal \mathbb{F}_3 -linear \mathbb{F}_{3^2} -codes of length 7 is $2 \cdot (3^{d_1/2} + 2) = 58$.

Step 6 By Theorem 4, \mathcal{C} is cyclic Δ -self-dual if and only if for each i ($0 \leq i \leq 1$), the following assertions hold.

- (i) When $i = 0$, then \mathcal{C}_0 is 1-dimensional \mathcal{K}_0 -subspace of \mathcal{J}_0 with basis

$$\{\rho_{0,0}(X)^2\} = \{\omega^2 X^6 + \omega^2 X^5 + \omega^2 X^4 + \omega^2 X^3 + \omega^2 X^2 + \omega^2 X + \omega^2\}.$$

(ii) When $i \neq 0$, we have $\mu(1) = 1$ and $\tau_{1,-1}(\mathcal{I}_{1,0}) = \mathcal{I}_{1,1}$. Then \mathcal{C}_1 is 1-dimensional over \mathcal{K}_1 with bases $\{e_{1,0}(X)\} = \{\omega^5 X^6 + \omega^5 X^5 + \omega^7 X^4 + \omega^5 X^3 + \omega^7 X^2 + \omega^7 X\}$, $\{e_{1,1}(X)\} = \{\omega^7 X^6 + \omega^7 X^5 + \omega^5 X^4 + \omega^7 X^3 + \omega^5 X^2 + \omega^5 X\}$ and $\{e_{1,0}(X) + \rho_{1,1}(X)^k\}$, where $k = 28m$ for $0 \leq m \leq 25$.

According to Theorem 5, the number of cyclic Δ -self-dual \mathbb{F}_3 -linear \mathbb{F}_{3^2} -codes of length 7 is $3^{d_1/2} + 1 = 28$.

A good code By Step 5, we have that $\mathcal{C} = \langle e_{1,0}(X) \rangle = \langle \omega^5 X^6 + \omega^5 X^5 + \omega^7 X^4 + \omega^5 X^3 + \omega^7 X^2 + \omega^7 X \rangle$ is a cyclic Δ -self-orthogonal \mathbb{F}_3 -linear \mathbb{F}_{3^2} -code of length 7. Using Magma, it is not difficult to show that \mathcal{C} is a $(7, (3^2)^3, 5)$ \mathbb{F}_3 -linear \mathbb{F}_{3^2} -code with generator matrix

$$\begin{pmatrix} 0 & w^7 & w^7 & w^5 & w^7 & w^5 & w^5 \\ w^5 & 0 & w^7 & w^7 & w^5 & w^7 & w^5 \\ w^5 & w^5 & 0 & w^7 & w^7 & w^5 & w^7 \\ w^7 & w^5 & w^5 & 0 & w^7 & w^7 & w^5 \\ w^5 & w^7 & w^5 & w^5 & 0 & w^7 & w^7 \\ w^7 & w^5 & w^7 & w^5 & w^5 & 0 & w^7 \end{pmatrix}.$$

Furthermore, \mathcal{C} is a good code which has the same parameters with the best known linear code $[7, 3, 5]$ over \mathbb{F}_{3^2} according to the online database [11].

At the end of this example, with the similar manner above, we construct several good cyclic Δ -self-orthogonal \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes of length n in Table 1. These codes have the same parameters with the best known linear codes over \mathbb{F}_{q^2} given in online database [11] or MDS codes over \mathbb{F}_{q^2} . In Table 1, n is the length of \mathcal{C} , $(q^2)^k$ is the cardinality of \mathcal{C} , d is the minimum Hamming distance of \mathcal{C} and α_i are given in Appendix A, where $1 \leq i \leq 15$.

Table 1 Some good \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes

$\{q, n\}$	The basis of \mathcal{C}	$(n, (q^2)^k, d)$
$\{2, 11\}$	α_1	$(11, (2^2)^5, 6)$
$\{2, 19\}$	α_2	$(19, (2^2)^9, 8)$
$\{3, 7\}$	α_3	$(7, (3^2)^3, 5)$
$\{3, 19\}$	α_4	$(19, (3^2)^9, 10)$
$\{5, 7\}$	α_5	$(7, (5^2)^3, 5)$
$\{5, 23\}$	α_6	$(23, (5^2)^{11}, 12)$
$\{7, 11\}$	α_7	$(11, (7^2)^5, 7)$
$\{7, 23\}$	α_8	$(23, (7^2)^{11}, 12)$
$\{11, 23\}$	α_9	$(23, (11^2)^{11}, 12)$
$\{13, 11\}$	α_{10}	$(11, (13^2)^5, 7)$
$\{13, 19\}$	α_{11}	$(19, (13^2)^9, 11)$
$\{17, 7\}$	α_{12}	$(7, (17^2)^3, 5)$
$\{17, 11\}$	α_{13}	$(11, (17^2)^5, 7)$
$\{19, 7\}$	α_{14}	$(7, (19^2)^3, 5)$
$\{19, 11\}$	α_{15}	$(11, (19^2)^5, 7)$

6 Conclusions

A new trace bilinear form on $\mathbb{F}_{q^t}^n$ which is called Δ -bilinear form is given, where n is a positive integer coprime to q . Then according to this new trace bilinear form, bases and enumeration of cyclic Δ -self-orthogonal and cyclic Δ -self-dual \mathbb{F}_q -linear \mathbb{F}_{q^t} -codes are investigated when $t = 2$. Finally, we describe a program to construct cyclic Δ -self-orthogonal and cyclic Δ -self-dual \mathbb{F}_3 -linear \mathbb{F}_{3^2} -codes of length 7 and obtain some good \mathbb{F}_q -linear \mathbb{F}_{q^2} -codes.

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Appendix A

In Table 1,

$$\alpha_1 = \omega_1 X^{10} + \omega_1^2 X^9 + \omega_1 X^8 + \omega_1 X^7 + \omega_1 X^6 + \omega_1^2 X^5 + \omega_1^2 X^4 + \omega_1^2 X^3 + \omega_1 X^2 + \omega_1^2 X + 1,$$

$$\alpha_2 = \omega_1 X^{18} + \omega_1^2 X^{17} + \omega_1^2 X^{16} + \omega_1 X^{15} + \omega_1 X^{14} + \omega_1 X^{13} + \omega_1 X^{12} + \omega_1^2 X^{11} + \omega_1 X^{10} + \omega_1^2 X^9 + \omega_1 X^8 + \omega_1^2 X^7 + \omega_1^2 X^6 + \omega_1^2 X^5 + \omega_1^2 X^4 + \omega_1 X^3 + \omega_1 X^2 + \omega_1^2 X + 1,$$

$$\alpha_3 = \omega^5 X^6 + \omega^5 X^5 + \omega^7 X^4 + \omega^5 X^3 + \omega^7 X^2 + \omega^7 X,$$

$$\alpha_4 = \omega^5 X^{18} + \omega^7 X^{17} + \omega^7 X^{16} + \omega^5 X^{15} + \omega^5 X^{14} + \omega^5 X^{13} + \omega^5 X^{12} + \omega^7 X^{11} + \omega^5 X^{10} + \omega^7 X^9 + \omega^5 X^8 + \omega^7 X^7 + \omega^7 X^6 + \omega^7 X^5 + \omega^7 X^4 + \omega^5 X^3 + \omega^5 X^2 + \omega^7 X,$$

$$\alpha_5 = \omega_2^7 X^6 + \omega_2^7 X^5 + \omega_2^{11} X^4 + \omega_2^7 X^3 + \omega_2^{11} X^2 + \omega_2^{11} X + 4,$$

$$\alpha_6 = \omega_2^{22} X^{22} + \omega_2^{22} X^{21} + \omega_2^{22} X^{20} + \omega_2^{22} X^{19} + \omega_2^{14} X^{18} + \omega_2^{22} X^{17} + \omega_2^{14} X^{16} + \omega_2^{22} X^{15} + \omega_2^{22} X^{14} + \omega_2^{14} X^{13} + \omega_2^{14} X^{12} + \omega_2^{22} X^{11} + \omega_2^{22} X^{10} + \omega_2^{14} X^9 + \omega_2^{14} X^8 + \omega_2^{22} X^7 + \omega_2^{14} X^6 + \omega_2^{22} X^5 + \omega_2^{14} X^4 + \omega_2^{14} X^3 + \omega_2^{14} X^2 + \omega_2^{14} X + 2,$$

$$\alpha_7 = \omega_3^{41} X^{10} + \omega_3^{47} X^9 + \omega_3^{41} X^8 + \omega_3^{41} X^7 + \omega_3^{41} X^6 + \omega_3^{47} X^5 + \omega_3^{47} X^4 + \omega_3^{47} X^3 + \omega_3^{41} X^2 + \omega_3^{47} X + 3,$$

$$\alpha_8 = \omega_3^{35} X^{22} + \omega_3^{35} X^{21} + \omega_3^{35} X^{20} + \omega_3^{35} X^{19} + \omega_3^5 X^{18} + \omega_3^{35} X^{17} + \omega_3^5 X^{16} + \omega_3^{35} X^{15} + \omega_3^{35} X^{14} + \omega_3^5 X^{13} + \omega_3^5 X^{12} + \omega_3^{35} X^{11} + \omega_3^{35} X^{10} + \omega_3^5 X^9 + \omega_3^5 X^8 + \omega_3^{35} X^7 + \omega_3^5 X^6 + \omega_3^{35} X^5 + \omega_3^5 X^4 + \omega_3^5 X^3 + \omega_3^5 X^2 + \omega_3^5 X + 2,$$

$$\alpha_9 = \omega_4^{99} X^{22} + \omega_4^{99} X^{21} + \omega_4^{99} X^{20} + \omega_4^{99} X^{19} + \omega_4^9 X^{18} + \omega_4^{99} X^{17} + \omega_4^9 X^{16} + \omega_4^{99} X^{15} + \omega_4^{99} X^{14} + \omega_4^9 X^{13} + \omega_4^9 X^{12} + \omega_4^{99} X^{11} + \omega_4^{99} X^{10} + \omega_4^9 X^9 + \omega_4^9 X^8 + \omega_4^{99} X^7 + \omega_4^9 X^6 + \omega_4^{99} X^5 + \omega_4^9 X^4 + \omega_4^9 X^3 + \omega_4^9 X^2 + \omega_4^9 X,$$

$$\alpha_{10} = \omega_5^{158} X^{10} + \omega_5^{38} X^9 + \omega_5^{158} X^8 + \omega_5^{158} X^7 + \omega_5^{158} X^6 + \omega_5^{38} X^5 + \omega_5^{38} X^4 + \omega_5^{38} X^3 + \omega_5^{158} X^2 + \omega_5^{38} X + 4,$$

$$\alpha_{11} = \omega_5^{143} X^{18} + \omega_5^{11} X^{17} + \omega_5^{11} X^{16} + \omega_5^{143} X^{15} + \omega_5^{143} X^{14} + \omega_5^{143} X^{13} + \omega_5^{143} X^{12} + \omega_5^{11} X^{11} + \omega_5^{143} X^{10} + \omega_5^{11} X^9 + \omega_5^{143} X^8 + \omega_5^{11} X^7 + \omega_5^{11} X^6 + \omega_5^{11} X^5 + \omega_5^{11} X^4 + \omega_5^{143} X^3 + \omega_5^{143} X^2 + \omega_5^{11} X + 8,$$

$$\alpha_{12} = \omega_6^{40} X^6 + \omega_6^{40} X^5 + \omega_6^{104} X^4 + \omega_6^{40} X^3 + \omega_6^{104} X^2 + \omega_6^{104} X + 15,$$

$$\alpha_{13} = \omega_6^{19} X^{10} + \omega_6^{35} X^9 + \omega_6^{19} X^8 + \omega_6^{19} X^7 + \omega_6^{19} X^6 + \omega_6^{35} X^5 + \omega_6^{35} X^4 + \omega_6^{35} X^3 + \omega_6^{19} X^2 + \omega_6^{35} X + 2,$$

$$\alpha_{14} = \omega_7^{61} Y^6 + \omega_7^{61} Y^5 + \omega_7^{79} Y^4 + \omega_7^{61} Y^3 + \omega_7^{79} Y^2 + \omega_7^{79} Y + 14,$$

$$\alpha_{15} = \omega_7^{331} X^{10} + \omega_7^{169} X^9 + \omega_7^{331} X^8 + \omega_7^{331} X^7 + \omega_7^{331} X^6 + \omega_7^{169} X^5 + \omega_7^{169} X^4 + \omega_7^{169} X^3 + \omega_7^{331} X^2 + \omega_7^{169} X + 16,$$

ω_1 is a primitive element of $\mathbb{F}_{2^2} = \mathbb{F}_2[X]/\langle X^2 + X + 1 \rangle$ with $\text{ord}(\omega_1) = 2^2 - 1 = 3$,

ω is a primitive element of $\mathbb{F}_{3^2} = \mathbb{F}_3[X]/\langle X^2 + 2X + 2 \rangle$ with $\text{ord}(\omega) = 3^2 - 1 = 8$,

ω_2 is a primitive element of $\mathbb{F}_{5^2} = \mathbb{F}_5[X]/\langle X^2 + 4X + 2 \rangle$ with $\text{ord}(\omega_2) = 5^2 - 1 = 24$,

ω_3 is a primitive element of $\mathbb{F}_{7^2} = \mathbb{F}_7[X]/\langle X^2 + 6X + 3 \rangle$ with $\text{ord}(\omega_3) = 7^2 - 1 = 48$,

ω_4 is a primitive element of $\mathbb{F}_{11^2} = \mathbb{F}_{11}[X]/\langle X^2 + 7X + 2 \rangle$ with $\text{ord}(\omega_4) = 11^2 - 1 = 120$,

ω_5 is a primitive element of $\mathbb{F}_{13^2} = \mathbb{F}_{13}[X]/\langle X^2 + 12X + 2 \rangle$ with $\text{ord}(\omega_5) = 13^2 - 1 = 168$,

ω_6 is a primitive element of $\mathbb{F}_{17^2} = \mathbb{F}_{17}[X]/\langle X^2 + 16X + 3 \rangle$ with $\text{ord}(\omega_6) = 17^2 - 1 = 288$

and ω_7 is a primitive element of $\mathbb{F}_{19^2} = \mathbb{F}_{19}[X]/\langle X^2 + 18X + 2 \rangle$ with $\text{ord}(\omega_7) = 19^2 - 1 = 360$.

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